

Lecture 19: Poisson Equation with Dirichlet and Neumann Boundary Conditions

1 Introduction

In this lecture, we continue our study of elliptic partial differential equations by considering the **Poisson equation**

$$\Delta u = f(x, y)$$

in a bounded domain $D \subset \mathbb{R}^2$, together with either **Dirichlet** or **Neumann** boundary conditions. Throughout, we assume that:

- D is an open and connected set,
- ∂D denotes the boundary of D ,
- f and boundary data are sufficiently smooth.

Recall that a set D is connected if any two points in D can be joined by a continuous curve lying entirely in D .

2 Poisson Equation with Dirichlet Boundary Conditions

2.1 Problem Statement

We consider the Dirichlet problem for the Poisson equation:

$$\Delta u = f(x, y), \quad (x, y) \in D, \tag{1}$$

$$u = g(x, y), \quad (x, y) \in \partial D. \tag{2}$$

Our goal is to find a function $u \in C^2(D) \cap C(\overline{D})$ satisfying (1)–(2).

2.2 Decomposition Method

Motivated by ordinary differential equations, we write

$$u = v + w,$$

where:

- v is a *particular solution* of the Poisson equation,
- w is a solution of the homogeneous Laplace equation.

That is,

$$\begin{aligned}\Delta v &= f(x, y) \quad \text{in } D, \\ \Delta w &= 0 \quad \text{in } D.\end{aligned}$$

The boundary condition for w becomes

$$w|_{\partial D} = g - v|_{\partial D}.$$

Thus, once a particular solution v is found, the remaining function w is determined by solving a **Dirichlet problem for Laplace's equation**.

2.3 Finding a Particular Solution

If $f(x, y)$ is a polynomial of degree n , then it is natural to seek a polynomial solution $v(x, y)$ of degree $n + 2$, since the Laplacian reduces degree by two.

Example. Consider

$$\Delta u = -2 \quad \text{in } (0, a) \times (0, b),$$

with homogeneous boundary conditions

$$u = 0 \quad \text{on } \partial D.$$

We seek a quadratic polynomial

$$v(x, y) = A + Bx + Cy + Dx^2 + Exy + Fy^2.$$

Computing,

$$\Delta v = v_{xx} + v_{yy} = 2D + 2F.$$

To satisfy $\Delta v = -2$, we choose

$$D = -1, \quad F = 0.$$

A convenient choice is

$$v(x, y) = -x^2 + Ax.$$

The constant A is chosen so that $v = 0$ on $x = 0$ and $x = a$, giving

$$v(x, y) = x(a - x).$$

2.4 Solving for the Harmonic Part

Let

$$w = u - v.$$

Then

$$\Delta w = 0,$$

with boundary conditions determined by $-v$.

In a rectangle $(0, a) \times (0, b)$, separation of variables gives

$$w(x, y) = \sum_{n=1}^{\infty} \left(A_n \cosh\left(\frac{n\pi y}{a}\right) + B_n \sinh\left(\frac{n\pi y}{a}\right) \right) \sin\left(\frac{n\pi x}{a}\right).$$

The coefficients A_n, B_n are determined using Fourier sine series expansions of the boundary data.

2.5 Final Solution

The solution of the Poisson–Dirichlet problem is

$$u(x, y) = v(x, y) + w(x, y),$$

where v is a particular solution and w is a harmonic function ensuring the boundary condition.

3 Neumann Boundary Value Problem

3.1 Problem Statement

We now consider the Neumann problem:

$$\Delta u = 0 \quad \text{in } D, \tag{3}$$

$$\frac{\partial u}{\partial n} = h \quad \text{on } \partial D. \tag{4}$$

For a rectangle $D = (0, a) \times (0, b)$, the boundary conditions may be written as:

$$\begin{aligned} u_x(0, y) &= f_1(y), & u_x(a, y) &= f_2(y), \\ u_y(x, 0) &= g_1(x), & u_y(x, b) &= g_2(x). \end{aligned}$$

3.2 Compatibility Condition

A necessary condition for existence is the **compatibility condition**:

$$\int_0^b (f_1(y) - f_2(y)) dy + \int_0^a (g_1(x) - g_2(x)) dx = 0.$$

This follows from Green's identity.

3.3 Decomposition Strategy

We again decompose

$$u = u_1 + u_2,$$

where:

- u_1 accounts for the y -boundary data,
- u_2 accounts for the x -boundary data.

Each subproblem satisfies Laplace's equation with partial Neumann data.

3.4 Eigenfunction Expansion

Following Granberg's method, we seek solutions of the form

$$u(x, y) = Y_0(y) + \sum_{n=1}^{\infty} X_n(x)Y_n(y),$$

where

$$X_n(x) = \cos\left(\frac{n\pi x}{a}\right)$$

solve the eigenvalue problem

$$X_n'' + \lambda_n X_n = 0, \quad X_n'(0) = X_n'(a) = 0.$$

This leads to ordinary differential equations for $Y_n(y)$:

$$Y_n'' - \left(\frac{n\pi}{a}\right)^2 Y_n = F_n(y),$$

where F_n depends on the boundary data.

3.5 Solution Structure

The general solution is

$$Y_n(y) = A_n \cosh\left(\frac{n\pi y}{a}\right) + B_n \sinh\left(\frac{n\pi y}{a}\right) + (\text{particular solution}).$$

Coefficients are determined using Fourier cosine series and boundary conditions.

4 Summary

- Poisson problems are reduced to Laplace problems using decomposition.
- Polynomial forcing functions admit polynomial particular solutions.
- Dirichlet problems are solved using separation of variables and Fourier series.
- Neumann problems require compatibility conditions for solvability.
- Eigenfunction expansions provide a systematic solution framework.

End of Lecture 19