Lecture 18: Separation of Variables for the Laplace Equation

1 Problem Setup

We consider the Laplace equation in two dimensions:

$$\Delta u = u_{xx} + u_{yy} = 0 \quad \text{in } \Omega,$$

where the domain is the rectangular region

$$\Omega = (0, a) \times (0, b).$$

1.1 Boundary Conditions

The boundary conditions are prescribed as follows:

$$\begin{aligned}
u(x, 0) &= f(x), & 0 \leq x \leq a, \\
u(x, b) &= 0, & 0 \leq x \leq a, \\
u(0, y) &= 0, & 0 \leq y \leq b, \\
u(a, y) &= 0, & 0 \leq y \leq b.
\end{aligned}$$

That is, the function is given on the bottom side of the rectangle and vanishes on the remaining three sides.

2 Remarks on Separation of Variables

- The method of separation of variables works naturally on rectangular domains.
- For circular domains, polar coordinates are used.
- For spherical domains in three dimensions, spherical coordinates are employed.
- In each case, a suitable change of variables converts the domain into a rectangular one.

3 Linearity and Decomposition of Boundary Data

The Laplace equation is linear. Hence, if

$$\Delta u_i = 0 \quad \text{in } \Omega$$

for $i = 1, 2, 3, 4$, then

$$u = u_1 + u_2 + u_3 + u_4$$

also satisfies $\Delta u = 0$.

Therefore, a general Dirichlet problem on a rectangle can be solved by decomposing the boundary data into four parts, each supported on one side of the boundary, solving the corresponding subproblems, and summing the solutions.

In this lecture, we focus on the case where only $u(x, 0) = f(x)$ is nonzero.

4 Separation of Variables

We seek a nontrivial separable solution of the form

$$u(x, y) = X(x)Y(y).$$

Substituting into $\Delta u = 0$ gives

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

Dividing by $X(x)Y(y)$,

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda,$$

where λ is a separation constant. This leads to the system

$$\begin{cases} X'' - \lambda X = 0, \\ Y'' + \lambda Y = 0. \end{cases}$$

5 The Eigenvalue Problem for $X(x)$

From the boundary conditions $u(0, y) = u(a, y) = 0$, we obtain

$$X(0) = X(a) = 0.$$

To obtain nontrivial solutions, we choose $\lambda = -\alpha^2$. Then

$$X'' + \alpha^2 X = 0, \quad X(0) = X(a) = 0.$$

This Sturm–Liouville problem has eigenvalues

$$\alpha_n = \frac{n\pi}{a}, \quad n = 1, 2, 3, \dots$$

with corresponding eigenfunctions

$$X_n(x) = \sin\left(\frac{n\pi x}{a}\right).$$

6 Solution of the Y-Equation

For each n, the equation for $Y_n(y)$ becomes

$$Y_n'' - \left(\frac{n\pi}{a}\right)^2 Y_n = 0.$$

The general solution is

$$Y_n(y) = A_n \sinh\left(\frac{n\pi}{a}(b - y)\right).$$

The boundary condition $u(x, b) = 0$ implies $Y_n(b) = 0$, which is satisfied by this choice.

7 Superposition of Solutions

Each separated solution

$$u_n(x, y) = \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi}{a}(b - y)\right)$$

satisfies $\Delta u = 0$.

By linearity, the general solution is

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi}{a}(b - y)\right).$$

8 Determination of Coefficients

Using the boundary condition $u(x, 0) = f(x)$, we obtain

$$f(x) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi x}{a}\right).$$

This is the Fourier sine series expansion of $f(x)$. Hence,

$$A_n = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx.$$

9 Final Solution

The formal solution to the boundary value problem is

$$\boxed{u(x, y) = \sum_{n=1}^{\infty} \left[\frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(\xi) \sin\left(\frac{n\pi \xi}{a}\right) d\xi\right] \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi}{a}(b - y)\right)}$$

10 Conclusion

The separation of variables method provides an explicit solution for the Laplace equation on rect-angular domains. For general Dirichlet data, the solution is obtained by decomposing the boundary conditions, solving the corresponding subproblems, and superposing the results.