Lecture 17: Maximum Principle for the Laplacian

Partial Differential Equations

1 Motivation: A Priori Estimates

Consider the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain.

In many situations, it is not possible to obtain explicit solutions. Instead, we ask:

- If a solution exists, what properties must it satisfy?
- Is the solution unique?
- Where do its maximum and minimum values occur?

Such results are called *a priori estimates*, since they are derived without knowing the explicit solution.

If $f \in C(\overline{\Omega})$ and u satisfies $-\Delta u = f$, then necessarily

$$\Delta u \in C(\overline{\Omega}), \quad \text{and hence } u \in C^2(\Omega).$$

2 Harmonic Functions

Definition 2.1. A function u is called *harmonic* in a domain D if

$$\Delta u = 0 \quad \text{in } D.$$

Examples

- $u(x, y) = Ax + By + C$
- $u(x, y) = x^2 - y^2$
- Constant functions

Throughout, we assume:

- $D \subset \mathbb{R}^2$ is open, connected, and bounded
- $u \in C^2(D) \cap C(\overline{D})$

Here, ∂D denotes the boundary of D, and $\overline{D} = D \cup \partial D$.

3 Maximum Principle

Theorem 3.1 (Maximum Principle)**.** *Let u be harmonic in a bounded domain $D \subset \mathbb{R}^2$ and continuous on $\overline{D}$. Then*

$$\max_{\overline{D}} u = \max_{\partial D} u.$$

In particular, a non-constant harmonic function cannot attain a maximum in the interior of D.

3.1 Physical Interpretation

If u represents the temperature in a body with no internal heat sources or sinks, then the hottest and coldest points occur on the surface of the body.

4 Proof Sketch of the Maximum Principle

Assume, for contradiction, that u attains a maximum m_0 at an interior point $(x_0, y_0) \in D$. Let

$$m = \max_{\partial D} u,$$

and assume $m_0 > m$.

Since D is bounded, it is contained in a ball of radius R centered at (x_0, y_0). Define

$$v(x,y) = u(x,y) + \frac{m_0 - m}{4R^2}\left[(x-x_0)^2 + (y-y_0)^2\right].$$

Then:

- $v(x_0, y_0) = m_0$
- $v < m_0$ on ∂D

Thus, v attains a maximum in the interior of D. At an interior maximum, the Hessian is negative semidefinite, implying

$$\Delta v \leq 0.$$

However,

$$\Delta v = \Delta u + \frac{m_0 - m}{R^2} = \frac{m_0 - m}{R^2} > 0,$$

which is a contradiction. Hence, the maximum must occur on the boundary.

5 Minimum Principle

Theorem 5.1 (Minimum Principle)**.** *Let u be harmonic in D and continuous on $\overline{D}$. Then*

$$\min_{\overline{D}} u = \min_{\partial D} u.$$

Proof. Apply the maximum principle to $-u$. □

6 Uniqueness of the Dirichlet Problem

Consider the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u = f & \text{on } \partial D. \end{cases} \tag{2}$$

Theorem 6.1 (Uniqueness)**.** *If a solution exists in $C^2(D) \cap C(\overline{D})$, then it is unique.*

Proof. Suppose u_1 and u_2 are two solutions and define $\phi = u_1 - u_2$. Then

$$\Delta \phi = 0 \quad \text{in } D, \qquad \phi = 0 \quad \text{on } \partial D.$$

By the maximum and minimum principles,

$$\max_{\overline{D}} \phi = \min_{\overline{D}} \phi = 0,$$

hence $\phi \equiv 0$ and $u_1 = u_2$. □

7 Extension to the Poisson Equation

For the Poisson problem

$$\begin{cases} -\Delta u = f & \text{in } D, \\ u = g & \text{on } \partial D, \end{cases}$$

any two solutions differ by a harmonic function vanishing on the boundary. Hence, the solution is unique if it exists.

8 Continuous Dependence on Boundary Data

Let u_1, u_2 satisfy

$$\Delta u_i = 0 \text{ in } D, \quad u_i = f_i \text{ on } \partial D.$$

Define $v = u_1 - u_2$. Then

$$\Delta v = 0 \text{ in } D, \quad v = f_1 - f_2 \text{ on } \partial D.$$

By the maximum principle,

$$\|u_1 - u_2\|_{L^\infty(D)} \leq \|f_1 - f_2\|_{L^\infty(\partial D)}.$$

Thus, the solution depends continuously on the boundary data.

9 Well-Posedness

Once existence is established, the Dirichlet problem for Laplace and Poisson equations is well-posed:

- Uniqueness
- Continuous dependence on boundary data

10 Advanced Remark

Let $\{u_n\}$ be harmonic in D and continuous on $\overline{D}$. If $u_n \to f$ uniformly on ∂D, then u_n converges uniformly in $\overline{D}$.

11 Summary

- Harmonic functions have no interior maxima or minima
- Extremal values occur on the boundary
- Maximum principle yields uniqueness and stability
- It is a fundamental tool in elliptic PDE theory