Lecture 16: Laplace Equation - Fundamental Solution

1 The Laplace and Poisson Equations

Let $\Omega \subset \mathbb{R}^n$ be a *domain*, i.e. an open and connected set.

Laplace Equation

$$\Delta u = 0 \quad \text{in } \Omega.$$

Poisson Equation

$$\Delta u = f \quad \text{in } \Omega,$$

where

- $u : \overline{\Omega} \to \mathbb{R}$ is the unknown function,
- $f : \overline{\Omega} \to \mathbb{R}$ is a given function.

The Laplacian of u is defined by

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

In many engineering texts, the Laplacian is written as $\nabla^2 u$. Although this is an abuse of notation, it is commonly used.

2 Harmonic Functions

A function $u \in C^2(\Omega)$ is called *harmonic* in Ω if

$$\Delta u = 0 \quad \text{in } \Omega.$$

Example

In $\mathbb{R}^2$, the function

$$u(x_1, x_2) = ax_1 + bx_2$$

is harmonic since all second derivatives vanish.

3 Physical Interpretation

Let $u(x)$ represent the density of a chemical concentration at equilibrium. For any smooth subregion $V \subset \Omega$, the net flux across ∂V is zero:

$$\int_{\partial V} \mathbf{F} \cdot \mathbf{n} \, ds = 0,$$

where $\mathbf{F}$ denotes the flux density and $\mathbf{n}$ is the outward unit normal.

Green–Gauss Theorem

$$\int_{V} \nabla \cdot \mathbf{F} \, dx = \int_{\partial V} \mathbf{F} \cdot \mathbf{n} \, ds.$$

Since the boundary integral vanishes for all V, we conclude that

$$\nabla \cdot \mathbf{F} = 0 \quad \text{in } \Omega.$$

4 Conservative Law

The flux is proportional to the negative gradient of the density:

$$\mathbf{F} = -c\nabla u, \quad c > 0.$$

Thus,

$$\nabla \cdot \mathbf{F} = -c\Delta u = 0,$$

which yields the Laplace equation

$$\Delta u = 0.$$

5 Applications

The Laplace equation appears in several physical models:

- Chemical diffusion (Fick's law),
- Heat conduction (Fourier's law),
- Electrostatics (Ohm's law).

6 Radial Solutions

We seek solutions of the Laplace equation in $\mathbb{R}^n$ that depend only on the radial variable

$$r = |x| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

A function u is *radial* if

$$u(x) = v(r).$$

7 Laplacian of a Radial Function

Let $u(x) = v(r)$. Then

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}, \quad \frac{\partial u}{\partial x_i} = v'(r)\frac{x_i}{r}.$$

The second derivatives satisfy

$$\frac{\partial^2 u}{\partial x_i^2} = v''(r)\frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right).$$

Summing over $i = 1, \ldots, n$, we obtain

$$\Delta u = v''(r) + \frac{n-1}{r}v'(r).$$

8 Reduced ODE

The Laplace equation $\Delta u = 0$ reduces to

$$v''(r) + \frac{n-1}{r}v'(r) = 0, \quad r > 0.$$

9 Solutions

Case $n = 2$

$$v(r) = C_1 \log r + C_2.$$

Case $n \geq 3$

$$v(r) = C_1 r^{2-n} + C_2.$$

10 Fundamental Solution

The *fundamental solution* of the Laplace equation in $\mathbb{R}^n$ is

$$\Phi(x) = \begin{cases} -\dfrac{1}{2\pi} \log |x|, & n = 2, \\ \dfrac{1}{n(n-2)\alpha_n} \dfrac{1}{|x|^{n-2}}, & n \geq 3, \end{cases} \qquad x \in \mathbb{R}^n \setminus \{0\},$$

where α_n denotes the surface area of the unit sphere in $\mathbb{R}^n$.

11 Remarks

- Φ is harmonic in $\mathbb{R}^n \setminus \{0\}$.
- The singularity at $x = 0$ corresponds to a point source.
- The fundamental solution is central to Green's functions and potential theory.

12 Conclusion

The Laplace equation models equilibrium phenomena across physics and engineering. Radial symmetry allows reduction of the PDE to an ODE, leading naturally to the fundamental solution, a cornerstone of elliptic partial differential equations.