Lecture 15: Elliptic Equations
Boundary Conditions

1 Elliptic Second-Order Equations

Consider a linear second-order partial differential equation in two variables:

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y = f.$$

The classification depends on the discriminant:

$$\Delta = b^2 - 4ac.$$

- $\Delta < 0$: Elliptic
- $\Delta = 0$: Parabolic
- $\Delta > 0$: Hyperbolic

In this lecture, we focus on **elliptic equations**.

2 Prototype: The Laplace Equation

The most important elliptic equation is the **Laplace equation**:

$$\Delta u = u_{xx} + u_{yy} = 0.$$

The operator Δ is called the **Laplacian**. In engineering literature, this is sometimes written as $\nabla^2 u$, which is only a symbolic notation.

2.1 Higher Dimensions

- In $\mathbb{R}^2$:
$$\Delta u = u_{xx} + u_{yy}$$
- In $\mathbb{R}^3$:
$$\Delta u = u_{xx} + u_{yy} + u_{zz}$$
- In $\mathbb{R}^n$:
$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i}$$

3 Related Elliptic Equations

3.1 Poisson Equation

$$\Delta u = g(x),$$

where g is a given function of position.

3.2 Helmholtz Equation (Eigenvalue Problem)

$$-\Delta u = \lambda u, \quad \lambda > 0.$$

This is the multidimensional analogue of the ODE eigenvalue problem

$$u'' + \lambda u = 0.$$

3.3 Steady-State Schrödinger Equation

$$-\Delta u + q(x)u = \lambda u.$$

4 Domains

A **domain** $D \subset \mathbb{R}^n$ is an open and connected set.

- **Open**: No boundary points included.
- **Connected**: Any two points in D can be joined by a continuous path lying entirely in D.

Examples:

- $\mathbb{R}^n$
- An open ball $B_1(0)$

5 Harmonic Functions

A function $u \in C^2(D)$ is called **harmonic** in D if

$$\Delta u = 0 \quad \text{in } D.$$

5.1 Examples

1. Constant functions: $u(x, y) = C$
2. Linear functions: $u(x, y) = ax + by + c$
3. Quadratic function:
$$u(x, y) = x^2 - y^2$$

In each case, $\Delta u = 0$.

6 Boundary Value Problems

Unlike ODEs, elliptic PDEs require **boundary data** to ensure uniqueness.

Let $D \subset \mathbb{R}^n$ be a domain with boundary ∂D.

7 Dirichlet Boundary Condition

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u = f(s) & \text{on } \partial D. \end{cases} \tag{1}$$

- f is a prescribed continuous function on ∂D.
- Specifies the value of the solution on the boundary.

7.1 Physical Interpretation

Represents the steady-state temperature distribution in a body with:

- No heat sources or sinks
- Temperature fixed on the boundary

8 Neumann Boundary Condition

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ \dfrac{\partial u}{\partial \eta} = f(s) & \text{on } \partial D. \end{cases} \tag{2}$$

Here, η denotes the outward unit normal vector to the boundary.

8.1 Compatibility Condition

For solvability, the following must hold:

$$\int_{\partial D} f(s)\, ds = 0.$$

8.2 Physical Interpretation

- $\frac{\partial u}{\partial \eta}$ represents heat flux
- The compatibility condition ensures zero net heat flow across the boundary

9 Mixed (Robin-Type) Boundary Condition

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ \dfrac{\partial u}{\partial \eta} + h(s)u = f(s) & \text{on } \partial D. \end{cases} \tag{3}$$

- h, f are continuous functions
- Models radiative heat exchange with the surroundings

10 Mixed Dirichlet-Neumann Boundary Condition

The boundary is decomposed as:

$$\partial D = \Gamma_1 \cup \Gamma_2, \quad \Gamma_1 \cap \Gamma_2 = \emptyset.$$

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u = f_1 & \text{on } \Gamma_1, \\ \dfrac{\partial u}{\partial \eta} = f_2 & \text{on } \Gamma_2. \end{cases} \tag{4}$$

- Dirichlet condition on part of the boundary
- Neumann condition on the remaining part

11 Summary

- Elliptic equations model steady-state phenomena
- Boundary conditions determine uniqueness
- Common boundary conditions:
  - Dirichlet
  - Neumann
  - Mixed (Robin)
  - Mixed Dirichlet–Neumann