Lecture 14: Canonical Form – Examples

Objective

In this lecture, we illustrate how to deduce the *canonical form* of second–order linear partial differential equations (PDEs) using explicit examples of

- hyperbolic equations,
- parabolic equations, and
- elliptic equations.

The procedure involves classification, determination of characteristic curves, suitable change of variables, and reduction to canonical form.

General Second–Order Linear PDE

A general second–order linear PDE in two variables has the form

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} + (\text{lower–order terms}) = 0.$$

Classification

The type of the equation is determined by the discriminant

$$\Delta = b^2 - 4ac.$$

Type	**Condition**
Hyperbolic	$\Delta > 0$
Parabolic	$\Delta = 0$
Elliptic	$\Delta < 0$

1 Example 1: Hyperbolic Equation

Given Equation

$$y^2 u_{xx} - x^2 u_{yy} = 0.$$

Classification

Here

$$a = y^2, \quad b = 0, \quad c = -x^2,$$

and

$$\Delta = -4(y^2)(-x^2) = 4x^2y^2 > 0$$

for $x \neq 0$ and $y \neq 0$. Hence, the equation is hyperbolic.

Characteristic Equations

The characteristic equation is

$$\frac{dy}{dx} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Substituting the coefficients gives

$$\frac{dy}{dx} = \pm\frac{x}{y}.$$

Characteristic Curves

Integrating, we obtain

$$y^2 - x^2 = C_1, \qquad y^2 + x^2 = C_2.$$

Change of Variables

Choose

$$\xi = \frac{y^2 - x^2}{2}, \qquad \eta = \frac{y^2 + x^2}{2}.$$

This transformation has a nonzero Jacobian away from the coordinate axes.

Canonical Form

Applying the chain rule and substituting into the original PDE, all second-order terms cancel, yielding the following equation involving only first-order derivatives:

$$u_{\xi\eta} = A(\xi, \eta)u_\xi + B(\xi, \eta)u_\eta.$$

This is the first canonical form of a hyperbolic equation.

2 Example 2: Parabolic Equation

Given Equation

$$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 0.$$

Classification

$$a = x^2, \quad b = 2xy, \quad c = y^2,$$

so

$$\Delta = (2xy)^2 - 4(x^2)(y^2) = 0.$$

Hence, the equation is parabolic everywhere.

Characteristic Equation

$$\frac{dy}{dx} = \frac{b}{2a} = \frac{y}{x}.$$

Characteristic Curves

Solving,

$$y = Cx,$$

which represents a family of straight lines through the origin.

Change of Variables

Choose

$$\xi = \frac{y}{x}, \qquad \eta = y.$$

The Jacobian of this transformation is nonzero for $y \neq 0$.

Canonical Form

Substitution into the PDE yields

$$y^2 u_{\eta\eta} = 0.$$

For $y \neq 0$, this reduces to

$$u_{\eta\eta} = 0.$$

General Solution

Integrating twice with respect to η,

$$u(\xi, \eta) = \eta F(\xi) + G(\xi),$$

where F and G are arbitrary twice continuously differentiable functions. Returning to the original variables,

$$u(x, y) = y\, F\left(\frac{y}{x}\right) + G\left(\frac{y}{x}\right).$$

3 Example 3: Elliptic Equation

Given Equation

$$u_{xx} + x^2 u_{yy} = 0.$$

Classification

$$a = 1, \quad b = 0, \quad c = x^2,$$

so

$$\Delta = -4x^2 < 0 \quad \text{for } x \neq 0.$$

Hence, the equation is elliptic in the region $x \neq 0$.

Characteristic Equations

$$\frac{dy}{dx} = \pm ix.$$

Characteristic Curves

Integrating,

$$2y \mp ix^2 = C.$$

Define

$$\xi = 2y - ix^2, \qquad \eta = 2y + ix^2.$$

Real Variables

Introduce real variables

$$\alpha = \frac{\xi + \eta}{2} = 2y, \qquad \beta = \frac{\xi - \eta}{2i} = -x^2.$$

The Jacobian determinant is

$$J = \begin{vmatrix} \alpha_x & \alpha_y \\ \beta_x & \beta_y \end{vmatrix} = 4x \neq 0 \quad (x \neq 0).$$

Canonical Form

After applying the chain rule and simplifying, the equation reduces to

$$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{2\beta} u_\beta,$$

which is the canonical form of the elliptic equation.

Conclusion

Through these examples, we see how suitable changes of variables reduce complicated second–order PDEs to simpler canonical forms, greatly facilitating analysis and solution.