Lecture 13: Canonical Forms of Second-Order Linear PDEs

1 General Second-Order Linear PDE

We consider a general second-order linear partial differential equation in two variables:

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y), \tag{1}$$

defined on a domain $\Omega \subset \mathbb{R}^2$.

Assumptions:

- The coefficients a, b, c, d, e, f, g are functions of (x, y).
- All coefficients are at least C^1 (continuously differentiable).
- The coefficients a, b, c are not simultaneously zero.

2 Discriminant and Classification

The *discriminant* is defined as

$$\Delta = b^2 - 4ac. \tag{2}$$

The nature of the PDE at a point (x_0, y_0) is determined by $\Delta(x_0, y_0)$:

- **Hyperbolic** if $\Delta > 0$
- **Parabolic** if $\Delta = 0$
- **Elliptic** if $\Delta < 0$

3 Change of Variables and Transformed Coefficients

Let (ξ, η) be new variables defined by

$$\xi = \xi(x,y), \qquad \eta = \eta(x,y),$$

with a non-vanishing Jacobian.

Under this transformation, the second-order terms become

$$A^* = a\xi_x^2 + b\xi_x\xi_y + c\xi_y^2, \tag{3}$$

$$B^* = 2a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + 2c\xi_y\eta_y, \tag{4}$$

$$C^* = a\eta_x^2 + b\eta_x\eta_y + c\eta_y^2. \tag{5}$$

Our goal is to choose ξ and η such that the equation reduces to a simpler *canonical form*.

4 Characteristic Equations

To eliminate the $u_{\xi\xi}$ and $u_{\eta\eta}$ terms, we require

$$A^* = 0, \qquad C^* = 0.$$

This leads to the characteristic equation

$$a\left(\frac{dy}{dx}\right)^2 - b\frac{dy}{dx} + c = 0. \tag{6}$$

The roots are

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{7}$$

These define families of curves called *characteristic curves.*

5 Hyperbolic Case ($\Delta > 0$)

5.1 Characteristics

When $b^2 - 4ac > 0$, there are two real and distinct characteristic families:

$$\xi(x, y) = \text{const}, \qquad \eta(x, y) = \text{const}.$$

5.2 First Canonical Form

With this choice,

$$A^* = 0, \qquad C^* = 0, \qquad B^* \neq 0,$$

and the PDE reduces to

$$u_{\xi\eta} = \text{lower-order terms}. \tag{8}$$

5.3 Second Canonical Form

Define new variables

$$\alpha = \xi + \eta, \qquad \beta = \xi - \eta.$$

Then the equation becomes

$$u_{\alpha\alpha} - u_{\beta\beta} = \text{lower-order terms}. \tag{9}$$

Prototype: Wave equation

$$u_{tt} - u_{xx} = 0.$$

6 Parabolic Case ($\Delta = 0$)

6.1 Characteristics

When $b^2 - 4ac = 0$, there is only one family of characteristics:

$$\xi(x, y) = \text{const}.$$

6.2 Choice of Variables

Choose η independent of ξ such that the Jacobian is nonzero.

With this choice:

$$A^* = 0, \qquad B^* = 0, \qquad C^* \neq 0.$$

6.3 Canonical Form

The PDE reduces to

$$u_{\eta\eta} = \text{lower-order terms}. \tag{10}$$

Prototype: Heat equation

$$u_t - u_{xx} = 0.$$

7 Elliptic Case ($\Delta < 0$)

7.1 Complex Characteristics

When $b^2 - 4ac < 0$, the characteristic roots are complex conjugates. Let

$$\xi = \alpha + i\beta, \qquad \eta = \alpha - i\beta,$$

where α, β are real.

7.2 Real Variables

Define

$$\alpha = \frac{\xi + \eta}{2}, \qquad \beta = \frac{\xi - \eta}{2i}.$$

7.3 Canonical Form

After transformation,

$$A^{**} = C^{**}, \qquad B^{**} = 0,$$

and the PDE becomes

$$u_{\alpha\alpha} + u_{\beta\beta} = \text{lower-order terms}. \tag{11}$$

Prototype: Laplace equation

$$u_{xx} + u_{yy} = 0.$$

8 Summary

Type	Discriminant	Canonical Form
Hyperbolic	$\Delta > 0$	$u_{\alpha\alpha} - u_{\beta\beta}$
Parabolic	$\Delta = 0$	$u_{\eta\eta}$
Elliptic	$\Delta < 0$	$u_{\alpha\alpha} + u_{\beta\beta}$

In the following lectures, we will work through explicit examples demonstrating how to compute these canonical forms in practice.