Lecture 01 - Part B
Fourier Series

Introduction

In this lecture, we introduce the basic idea of Fourier series and explain how a periodic function can be represented using trigonometric functions.

Motivation

The main question we want to address is the following:

> *Given a 2π-periodic function $f(x)$, can we represent it using simpler, well–understood functions?*

The reason for focusing on 2π-periodic functions will become clear in later lectures. For now, we restrict ourselves to this class of periodic functions.

Recall from the previous lecture that if f and g are p-periodic functions, then any linear combination

$$af(x) + bg(x),$$

where $a, b \in \mathbb{R}$, is also p-periodic. Therefore, to study 2π-periodic functions, it is natural to consider linear combinations of functions that are themselves 2π-periodic.

The Trigonometric System

Some basic examples of 2π-periodic functions are

$$1, \quad \sin x, \quad \cos x, \quad \sin 2x, \quad \cos 2x, \quad \ldots, \quad \sin(nx), \quad \cos(nx), \ \ldots$$

The collection of functions

$$\{1,\ \cos(nx),\ \sin(nx) \mid n \in \mathbb{N}\}$$

is called the *trigonometric system*. This is merely a name for the set and does not imply any additional structure at this stage.

Our goal is to represent a given 2π-periodic function $f(x)$ using elements of this trigonometric system.

Trigonometric Series

A series formed using the trigonometric system is called a *trigonometric series*. A general trigonometric series has the form

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$

Using summation notation, this can be written as

$$a_0 + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big),$$

where a_n and b_n are real constants.

Fourier Series Representation

The central question is whether a given 2π-periodic function $f(x)$ can be represented by such a trigonometric series. That is, can we write

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big)? \qquad (\star)$$

Since the right-hand side is an infinite series, issues of convergence naturally arise. Even when the series converges, the equality in $(\star)$ may not hold for all x. These questions will be studied in detail in later lectures.

If the series in $(\star)$ converges to $f(x)$ (in an appropriate sense), then it is called the *Fourier series representation* of f.

The constants

$$a_0,\ a_n,\ b_n \quad (n \in \mathbb{N})$$

are called the *Fourier coefficients* of the function f.

Euler–Fourier Coefficients

A natural question is whether the Fourier coefficients can be computed directly from the function $f(x)$. The answer is yes, and the formulas are known as *Euler's formulas*.

The coefficients are given by:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx, \quad n \in \mathbb{N},$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx, \quad n \in \mathbb{N}.$$

There is no coefficient b_0, since $\sin(0x) = 0$.

These formulas allow us to compute all Fourier coefficients explicitly once the function f is known.

Historical Origin

The theory of Fourier series originated from the work of the French mathematician *Jean-Baptiste Joseph Fourier* in the early 19th century. Fourier introduced this idea while studying the *heat equation* and problems related to heat flow.

He proposed that many functions arising in physical problems could be represented as infinite sums of simple trigonometric functions. This revolutionary idea laid the foundation for modern harmonic analysis and partial differential equations.

This concludes the introductory discussion on the idea of Fourier series.