Lecture 12: Classification of Second-Order Linear Partial Differential Equations

1 Introduction

In this lecture, we begin the study of *second-order partial differential equations (PDEs)*. Although second-order PDEs may be linear or nonlinear, we restrict our attention to *linear* equations.

The reason for this restriction is that nonlinear PDEs are generally difficult to study directly. However, many nonlinear equations can be approximated by linear equations through a process known as *linearization*. Thus, understanding linear PDEs is essential.

Throughout this lecture, we focus on equations in *two independent variables*.

2 General Linear Second-Order PDE

A general linear second-order PDE in n variables can be written as

$$\sum_{i,j=1}^{n} a_{ij}(x)\,\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\,\frac{\partial u}{\partial x_i} + f(x)\,u = g(x), \tag{1}$$

where

- $u = u(x_1, \dots, x_n)$ is the unknown function,
- $a_{ij} = a_{ji}$ (the coefficient matrix is symmetric),
- a_{ij}, b_i, f, g are real-valued smooth functions,
- at least one a_{ij} is nonzero.

The equation is called *second order* because the highest-order derivatives are of order two.

3 Two-Variable Case

In this course, we restrict ourselves to two independent variables (x, y). Then (1) reduces to

$$a(x,y)\,u_{xx} + b(x,y)\,u_{xy} + c(x,y)\,u_{yy} + d(x,y)\,u_x + e(x,y)\,u_y + f(x,y)\,u = g(x,y). \tag{2}$$

This is the *general form of a linear second-order PDE in two variables.*

4 Definition of Solution

Let $\Omega \subset \mathbb{R}^2$ be an open (and connected) set.

A function u is called a *solution* of (2) in Ω if

$$u \in C^2(\Omega)$$

and the equation holds for all $(x, y) \in \Omega$.

5 Prototype Examples

5.1 Laplace Equation (Elliptic)

$$u_{xx} + u_{yy} = 0.$$

Here,

$$a = 1, \quad b = 0, \quad c = 1.$$

5.2 Heat Equation (Parabolic)

One-dimensional heat equation:

$$u_t - u_{xx} = 0.$$

This is a second-order linear PDE in two variables (x, t).

5.3 Wave Equation (Hyperbolic)

One-dimensional wave equation:

$$u_{tt} - u_{xx} = 0.$$

6 Classification of Second-Order PDEs

The classification depends *only on the highest-order terms.* Consider the equation

$$a\, u_{xx} + b\, u_{xy} + c\, u_{yy} + \text{lower-order terms} = 0. \tag{3}$$

Define the *discriminant*

$$\Delta(x, y) = b^2 - 4ac.$$

6.1 Elliptic Equation

The equation is *elliptic at* (x_0, y_0) if

$$\Delta(x_0, y_0) < 0.$$

6.2 Parabolic Equation

The equation is *parabolic at* (x_0, y_0) if

$$\Delta(x_0, y_0) = 0.$$

6.3 Hyperbolic Equation

The equation is *hyperbolic at* (x_0, y_0) if

$$\Delta(x_0, y_0) > 0.$$

7 Examples of Classification

7.1 Laplace Equation

$$u_{xx} + u_{yy} = 0.$$

$$\Delta = 0 - 4(1)(1) = -4 < 0.$$

Hence, the Laplace equation is *elliptic*.

7.2 Heat Equation

$$u_t - u_{xx} = 0.$$

$$\Delta = 0.$$

Hence, it is *parabolic*.

7.3 Wave Equation

$$u_{tt} - u_{xx} = 0.$$

$$\Delta = 0 - 4(-1)(1) = 4 > 0.$$

Hence, it is *hyperbolic*.

7.4 Variable-Coefficient Example

Consider

$$x^2 u_{xx} + u_{xy} + y^2 u_{yy} = 0.$$

Here,

$$a = x^2, \quad b = 1, \quad c = y^2.$$

The discriminant is

$$\Delta = 1 - 4x^2 y^2.$$

Thus:

- Elliptic where $4x^2y^2 > 1$,
- Parabolic where $4x^2y^2 = 1$,
- Hyperbolic where $4x^2y^2 < 1$.

The classification may vary from point to point.

8 Change of Variables

To simplify equations, we introduce new variables

$$\xi = \xi(x, y), \qquad \eta = \eta(x, y),$$

with $\xi, \eta \in C^1$.

The change of variables is valid if the Jacobian

$$J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0$$

in the region of interest.

9 Effect on Classification

Under a valid change of variables,

$$\Delta^* = J^2 \Delta.$$

Since $J^2 > 0$, the sign of Δ is preserved. Therefore:

- elliptic equations remain elliptic,
- parabolic equations remain parabolic,
- hyperbolic equations remain hyperbolic.

10 Conclusion

Every second-order linear PDE can be locally reduced, via a suitable change of variables, to one of three canonical forms:

- Laplace equation (elliptic),
- Heat equation (parabolic),
- Wave equation (hyperbolic).

In the next lecture, we will study these *canonical forms* and how to derive them.