Lecture 11: Method of Characteristics
Existence and Uniqueness

1 Introduction

In this lecture, we complete the discussion of the *method of characteristics* by focusing on the *Cauchy problem* for first-order partial differential equations. Our main goal is to understand the conditions under which solutions *exist* and are *unique*. The key idea is the notion of *non-characteristic (transversal) initial data.*

2 A Model Problem

Consider the linear transport equation

$$u_t + a\,u_x = 0, \qquad a \in \mathbb{R}. \tag{1}$$

The initial condition is prescribed as

$$u(x,0) = \varphi(x), \tag{2}$$

where φ is a given smooth function.

2.1 General Solution

From the method of characteristics, the general solution of

$$u_t + au_x = 0$$

is

$$u(x,t) = f(x - at), \tag{3}$$

where $f \in C^\infty(\mathbb{R})$ is arbitrary.

Imposing the initial condition yields

$$f(x) = \varphi(x),$$

and therefore the unique solution is

$$\boxed{u(x,t) = \varphi(x - at).} \tag{4}$$

3 Geometric Interpretation

3.1 Initial Data Curve

The initial data are prescribed on the curve

$$\gamma = \{(x, 0) : x \in \mathbb{R}\} \subset \mathbb{R}^2.$$

In $\mathbb{R}^3$, the lifted initial curve (graph of the data) is

$$(x, 0, \varphi(x)).$$

This curve lies on the solution surface $z = u(x, t)$.

3.2 Characteristic Curves

The characteristic system associated with $u_t + au_x = 0$ is

$$\frac{dx}{ds} = a, \qquad \frac{dt}{ds} = 1, \qquad \frac{dz}{ds} = 0. \tag{5}$$

Solving with initial conditions

$$x(0) = r, \quad t(0) = 0, \quad z(0) = \varphi(r),$$

we obtain

$$x = r + as, \qquad t = s, \qquad z = \varphi(r).$$

Eliminating r and s, we recover

$$u(x, t) = \varphi(x - at).$$

Thus:

- Characteristics are straight lines $x - at =$ constant.
- The solution is constant along characteristic curves.

4 The General Cauchy Problem

Consider a first-order semi-linear PDE

$$a(x, y)u_x + b(x, y)u_y = c(x, y), \tag{6}$$

with Cauchy data

$$u|_\gamma = \varphi, \tag{7}$$

where γ is a curve in the (x, y)-plane.

The central questions are:

- Does a solution exist?
- Is the solution unique?

5 Characteristic and Non-Characteristic Data

5.1 Projected Characteristics

Projected characteristic curves satisfy

$$\frac{dx}{ds} = a(x, y), \qquad \frac{dy}{ds} = b(x, y). \tag{8}$$

5.2 Non-Characteristic Curve

Let the data curve be parameterized as

$$\gamma(r) = (\gamma_1(r), \gamma_2(r)).$$

The curve γ is called *non-characteristic* if

$$\boxed{a(\gamma(r))\, \gamma_2'(r) - b(\gamma(r))\, \gamma_1'(r) \neq 0 \quad \text{for all } r.} \tag{9}$$

This condition is also known as the *transversality condition*.

6 An Ill-Posed Example

Consider again

$$u_t + a u_x = 0,$$

with data prescribed on

$$\gamma : x - at = 0.$$

Since the characteristics are also given by $x - at =$ constant, the data curve itself is a characteristic. Because solutions are constant along characteristics, the initial data must be constant along γ.

Hence, for general φ, the problem is ill-posed:

$$\boxed{\text{Existence or uniqueness fails.}}$$

7 Quasi-Linear Case

Consider the quasi-linear equation

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u). \tag{10}$$

Cauchy data are prescribed by a curve in $\mathbb{R}^3$:

$$(x, y, u) = (F(r), G(r), H(r)), \quad r \in I.$$

The non-characteristic condition becomes

$$\boxed{a(F, G, H)\, G'(r) - b(F, G, H)\, F'(r) \neq 0.} \tag{11}$$

8 Existence and Uniqueness Theorem (Local)

[Existence and Uniqueness] Let $\Omega \subset \mathbb{R}^3$ be open and connected. Assume:

1. $a, b, c \in C^1(\Omega)$,
2. the initial curve $(F(r), G(r), H(r))$ is regular:
$$(F'(r))^2 + (G'(r))^2 \neq 0,$$
3. the transversality condition holds at a point
$$(x_0, y_0, u_0) = (F(r_0), G(r_0), H(r_0)).$$

Then the Cauchy problem admits a *unique solution* defined in a neighborhood of (x_0, y_0) in the (x, y)-plane.

9 Interpretation

- Existence and uniqueness are local results.
- Non-characteristic initial data are essential.
- Characteristic initial curves generally lead to ill-posed problems.

10 Summary

- Solutions propagate along characteristic curves.
- Initial data must be transversal to characteristics.
- The transversality condition ensures well-posedness.
- This completes the method of characteristics for first-order PDEs.

End of Lecture 11