Lecture 10: Separation of Variables for First-Order PDEs

1 Introduction

In this lecture, we study the method of *separation of variables* for solving first-order partial differential equations (PDEs). This method is particularly useful because it is simple, systematic, and often leads to explicit solutions. Moreover, it can be applied not only to linear equations but also to certain nonlinear first-order PDEs.

The main idea is to seek solutions that can be written either as a product or as a sum of functions of single variables.

2 General First-Order PDE

Consider a general first-order partial differential equation of the form

$$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y,u), \tag{1}$$

where $u = u(x,y)$.

A function u is called a classical solution of (1) in a domain $\Omega \subset \mathbb{R}^2$ if

$$u \in C^1(\Omega),$$

and equation (1) is satisfied for all $(x,y) \in \Omega$.

3 Separation of Variables: Basic Idea

We look for special solutions of the form:

- **Product form:**

$$u(x,y) = X(x)Y(y),$$

- **Sum form:**

$$u(x, y) = X(x) + Y(y).$$

Which form works depends on the equation. In practice, the product form is more commonly effective. We assume that X and Y are at least C^1 functions and not identically zero.

4 Example 1: Linear First-Order PDE

Consider the PDE

$$u_x + 2u_y = 0, \tag{1}$$

with the initial condition

$$u(0, y) = 4e^{-2y}. \tag{2}$$

4.1 Product Assumption

Assume

$$u(x, y) = X(x)Y(y).$$

Then

$$u_x = X'(x)Y(y), \qquad u_y = X(x)Y'(y).$$

Substituting into the PDE gives

$$X'(x)Y(y) + 2X(x)Y'(y) = 0.$$

Dividing by $X(x)Y(y)$, we obtain

$$\frac{X'(x)}{2X(x)} = -\frac{Y'(y)}{Y(y)}.$$

4.2 Separation Constant

Since the left-hand side depends only on x and the right-hand side only on y, both must be equal to a constant λ:

$$\frac{X'}{2X} = \lambda, \qquad -\frac{Y'}{Y} = \lambda.$$

4.3 Solving the ODEs

The resulting ordinary differential equations are

$$X'(x) = 2\lambda X(x), \qquad Y'(y) = -\lambda Y(y).$$

Their solutions are

$$X(x) = Ae^{2\lambda x}, \qquad Y(y) = Be^{-\lambda y}.$$

Hence,

$$u(x, y) = Ce^{2\lambda x - \lambda y}, \qquad C = AB.$$

4.4 Applying the Initial Condition

Using $u(0, y) = 4e^{-2y}$, we obtain

$$Ce^{-\lambda y} = 4e^{-2y}.$$

Thus,

$$\lambda = 2, \qquad C = 4.$$

4.5 Final Solution

$$\boxed{u(x, y) = 4e^{4x-2y}.}$$

5 Example 2: Nonlinear First-Order PDE

Consider the nonlinear PDE

$$y^2(u_x)^2 + x^2(u_y)^2 = x^2y^2u^2. \tag{2}$$

with $u(x, 0) = 3\exp\left(\frac{x^2}{4}\right)$.

5.1 Product Assumption

Assume

$$u(x, y) = X(x)Y(y).$$

Then

$$u_x = X'(x)Y(y), \qquad u_y = X(x)Y'(y).$$

Substituting into (2) gives

$$y^2[X'(x)]^2Y^2 + x^2X^2[Y'(y)]^2 = x^2y^2X^2Y^2.$$

Dividing by $x^2y^2X^2Y^2$, we obtain

$$\frac{1}{x^2}\left(\frac{X'}{X}\right)^2 + \frac{1}{y^2}\left(\frac{Y'}{Y}\right)^2 = 1.$$

5.2 Separation Constant

Rewriting,

$$\frac{1}{x^2}\left(\frac{X'}{X}\right)^2 = 1 - \frac{1}{y^2}\left(\frac{Y'}{Y}\right)^2 = \lambda^2.$$

5.3 Solving the ODEs

The resulting equations are

$$\frac{X'}{X} = \lambda x, \qquad \frac{Y'}{Y} = \sqrt{1-\lambda^2}\, y.$$

Thus,

$$X(x) = Ae^{\frac{\lambda}{2}x^2}, \qquad Y(y) = Be^{\frac{\sqrt{1-\lambda^2}}{2}y^2}.$$

5.4 General Solution

$$\boxed{u(x,y) = C\exp\left(\frac{\lambda}{2}x^2 + \frac{\sqrt{1-\lambda^2}}{2}y^2\right),} \quad C = AB.$$

Put the initial condition now and get the value of $C = 3$, $\lambda = \frac{1}{2}$. So $u(x,y) = 3\exp\left[\frac{1}{4}(x^2 + \sqrt{3}y^2)\right]$.

6 Example 3: Sum Assumption

Consider the equation

$$(u_x)^2 + (u_y)^2 = 1. \tag{3}$$

6.1 Sum Assumption

Assume

$$u(x,y) = X(x) + Y(y).$$

Then

$$u_x = X'(x), \qquad u_y = Y'(y).$$

Substituting into (3) yields

$$[X'(x)]^2 + [Y'(y)]^2 = 1.$$

6.2 Separation Constant

Thus,

$$[X'(x)]^2 = \lambda^2, \qquad [Y'(y)]^2 = 1 - \lambda^2.$$

6.3 General Solution

Solving,

$$X(x) = \lambda x + C_1, \qquad Y(y) = \sqrt{1-\lambda^2}\, y + C_2.$$

Hence, the general solution is

$$\boxed{u(x,y) = \lambda x + \sqrt{1-\lambda^2}\, y + C,}$$

where $C = C_1 + C_2$.

7 Conclusion

The method of separation of variables transforms a first-order PDE into ordinary differential equations. By choosing an appropriate assumption (product or sum), both linear and nonlinear equations can be solved efficiently. Initial or boundary conditions then determine the constants involved.