Lecture 09: Canonical Form for First-Order PDE

Introduction

In this lecture, we continue our study of first-order partial differential equations (PDEs) and discuss the canonical form for simplifying their solutions.

Example 1

Consider the PDE

$$(y-u)u_x + (u-x)u_y = x - y,$$

with the initial condition (Cauchy data)

$$u = 0 \quad \text{on} \quad xy = 1.$$

Characteristic Equations

The characteristic equations are:

$$\frac{dx}{y-u} = \frac{dy}{u-x} = \frac{du}{x-y}.$$

Parametric form:

$$\frac{dx}{dt} = y - u, \quad \frac{dy}{dt} = u - x, \quad \frac{du}{dt} = x - y.$$

Observation:

$$\frac{d}{dt}(x+y+u) = 0 \implies x + y + u = C_1,$$

$$\frac{d}{dt}(x^2+y^2+u^2) = 0 \implies x^2 + y^2 + u^2 = C_2.$$

Using Cauchy Data

Using the initial curve $xy = 1$ and $u = 0$, we find

$$C_1^2 = C_2 + 2.$$

Hence, the integral surface is given by

$$x + y + u = C_1, \quad x^2 + y^2 + u^2 = C_2, \quad C_1^2 = C_2 + 2.$$

Solving, we obtain

$$u(x, y) = \frac{1 - xy}{x + y}.$$

Example 2: Quasi-Linear Equation

Consider

$$uu_x + u_y = 1$$

with an initial curve parametrized by

$$x = x_0(s), \quad y = y_0(s), \quad u = u_0(s).$$

Parametric Form

Characteristic equations:

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = 1, \quad \frac{du}{dt} = 1.$$

With initial data:

$$x(s, 0) = x_0(s), \quad y(s, 0) = y_0(s), \quad u(s, 0) = u_0(s),$$

the solution of the system of ODEs gives a two-parameter family representing the integral surface.

Specific Initial Data

Let

$$x_0(s) = 2s^2, \quad y_0(s) = 2s, \quad u_0(s) = 0.$$

Solving, we get

$$x(s,t) = \frac{t^2}{2} + tu_0(s) + x_0(s), \quad y(s,t) = t + y_0(s), \quad u(s,t) = t + u_0(s).$$

Eliminating s and t gives the Cartesian form:

$$2u = y \pm \sqrt{4x - y^2},$$

valid for $y^2 < 4x$. The solution satisfies $u = 0$ on $y^2 = 2x$.

Canonical Form of a First-Order Linear PDE

Consider the general linear PDE

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y).$$

Transformation to New Coordinates

Introduce new variables:

$$\xi = \xi(x,y), \quad \eta = \eta(x,y),$$

with Jacobian

$$J = \frac{\partial(\xi,\eta)}{\partial(x,y)} = \xi_x\eta_y - \eta_x\xi_y \neq 0.$$

By the chain rule:

$$u_x = u_\xi\xi_x + u_\eta\eta_x, \quad u_y = u_\xi\xi_y + u_\eta\eta_y.$$

Substituting into the PDE:

$$Au_\xi + Bu_\eta + Cu = D,$$

where

$$A = a\xi_x + b\xi_y, \quad B = a\eta_x + b\eta_y, \quad C = c, \quad D = d.$$

Choosing Canonical Coordinates

Choose η such that

$$B = a\eta_x + b\eta_y = 0.$$

Then the PDE reduces to

$$u_\xi + \tilde{C}u = \tilde{D},$$

which is much easier to solve. This form is called the **canonical form**.

Example: Canonical Form

Consider

$$u_x - u_y = u.$$

Characteristic equations:

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{du}{u}.$$

Characteristic curves:

$$x + y = c_1, \quad y = c_2.$$

Choose

$$\xi = x + y, \quad \eta = y.$$

Then

$$u_x = u_\xi, \quad u_y = -u_\xi + u_\eta,$$

and the PDE reduces to

$$u_\eta = u,$$

which can be easily solved.

Conclusion

The canonical form simplifies first-order PDEs by an appropriate choice of new variables (ξ, η), typically chosen along characteristic curves. Once transformed, the PDE becomes easier to integrate and analyze.