Lecture 08: Geometry of First-Order PDE and Method of Characteristics

1 Introduction

In this lecture, we discuss the general first-order partial differential equation (PDE) and the method of characteristics. We start with a quasi-linear first-order PDE of the form:

$$a(x, y, u)u_x + b(x, y, u)u_y - c(x, y, u) = 0, \tag{1}$$

where $u = u(x, y)$ is the unknown function. Here, a, b, and c are smooth functions (at least C^1) from $\mathbb{R}^3$ to $\mathbb{R}$.

1.1 Definition of Solution

A function $u \in C^1(D)$, where $D \subset \mathbb{R}^2$, is a *solution* of (1) if

$$a(x, y, u)u_x + b(x, y, u)u_y - c(x, y, u) = 0 \quad \text{for all } (x, y) \in D.$$

We assume that u_x and u_y are continuously differentiable, ensuring the PDE is well-defined.

2 Geometric Interpretation

Let S be the surface corresponding to the graph of u, i.e.,

$$S = \{(x, y, z) \in \mathbb{R}^3 : z = u(x, y)\}.$$

Any point on S is $(x, y, u(x, y))$. The gradient vector of the function $F(x, y, z) = u(x, y) - z$ is

$$\nabla F = (u_x, u_y, -1),$$

which is normal to the surface S at each point.

The PDE (1) can be rewritten as the dot product

$$(a, b, c) \cdot \nabla F = 0,$$

which implies that the vector field (a, b, c) lies in the tangent plane of the surface S at each point. This vector field is called the *characteristic direction field*, and curves tangent to this field are called *characteristic curves*.

3 Characteristic Curves

Let C be a curve on S, parameterized as

$$C : (x(t), y(t), z(t)), \quad t \in \mathbb{R}.$$

The curve C satisfies

$$x'(t) = a(x, y, u), \quad y'(t) = b(x, y, u), \quad z'(t) = c(x, y, u),$$

which are called the *characteristic equations* of the PDE.

4 Method of Characteristics

The general solution of (1) can be expressed as

$$f(\phi(x, y, u), \psi(x, y, u)) = 0,$$

where ϕ and ψ are functions constant along characteristic curves. That is, along a characteristic curve:

$$\phi(x(t), y(t), u(t)) = C_1, \quad \psi(x(t), y(t), u(t)) = C_2.$$

From the characteristic equations, we have

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}.$$

4.1 Example 1

Consider the PDE:

$$xu_x + yu_y = u.$$

The characteristic equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u}.$$

Solving the first two equations:

$$\frac{dy}{dx} = \frac{y}{x} \implies \frac{y}{x} = C_1.$$

From the first and third:

$$\frac{du}{dx} = \frac{u}{x} \implies \frac{u}{x} = C_2.$$

Hence, the general solution is

$$f\left(\frac{y}{x}, \frac{u}{x}\right) = 0,$$

where f is an arbitrary smooth function.

4.2 Particular Solution

For example, choosing $f(w_1, w_2) = w_2$, we get

$$\frac{u}{x} = 0 \implies u = 0.$$

Alternatively, $f(w_1, w_2) = w_1 + w_2$ gives

$$\frac{y}{x} + \frac{u}{x} = 0 \implies u = -y.$$

4.3 Example 2

Consider

$$x^2 u_x + y^2 u_y = (x + y)u.$$

The characteristic equations are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{du}{(x + y)u}.$$

Solving the first two:

$$\frac{1}{x} - \frac{1}{y} = C_1.$$

From the first and third equations:

$$\frac{xy}{u} = C_2.$$

Thus, the general solution is

$$f\left(\frac{1}{x} - \frac{1}{y}, \frac{xy}{u}\right) = 0,$$

with f an arbitrary smooth function.

5 Cauchy Problem (Initial Value Problem)

Suppose a curve C in the xy-plane is parameterized as

$$x_0(t), y_0(t), \quad t \in I,$$

where $x_0'(t)^2 + y_0'(t)^2 \neq 0$. Given initial data $u_0(t)$ along C, there exists a solution $u(x, y)$ defined in a domain D containing C such that

$$u(x_0(t), y_0(t)) = u_0(t), \quad t \in I.$$

6 Remarks

- The geometric interpretation can be generalized to higher-order PDEs, but visualization beyond 3 dimensions is difficult.
- Nonlinear first-order PDEs have more complicated geometric interpretations (e.g., Monge cones).
- The method of characteristics is due to Lagrange and is particularly effective for first-order PDEs.