Lecture 07: First Order PDEs
Classification and Construction

1 General Form of a First-Order PDE

A general first-order partial differential equation (PDE) in two variables can be written as

$$F(x, y, u, u_x, u_y) = 0,$$

where $u = u(x, y)$ is an unknown function and

$$u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y}.$$

We assume that at least one of the partial derivatives appears non-trivially:

$$\frac{\partial F}{\partial u_x} \neq 0 \quad \text{or} \quad \frac{\partial F}{\partial u_y} \neq 0,$$

so that the equation genuinely involves first-order derivatives.

2 Linear and Nonlinear First-Order PDEs

2.1 Linear Equations

A linear first-order PDE has the form

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = d(x, y),$$

where a, b, c, d are given functions of x and y only.

2.2 Nonlinear Equations

If the equation involves nonlinear dependence on u, u_x, or u_y, then it is called nonlinear. For example,

$$u\, u_x + u_y = 0$$

or

$$u_x + (u_y)^2 = 0.$$

3 Classification of First-Order PDEs

First-order PDEs can be classified according to their dependence on u, u_x, and u_y.

3.1 Semilinear Equations

A first-order PDE is called *semilinear* if it is linear in the highest-order derivatives:

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u),$$

where c may depend nonlinearly on u, but u_x and u_y appear linearly.

We assume

$$a(x,y)^2 + b(x,y)^2 \neq 0.$$

Example

$$y\,u_x + x^2 u_y = u^2 + x^2 + y^2$$

is a semilinear equation.

3.2 Quasilinear Equations

A first-order PDE is called *quasilinear* if it is linear in the derivatives, but the coefficients may depend on u:

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u),$$

with

$$a^2 + b^2 \neq 0.$$

Example

$$u_x + u^2 u_y = x^2 + y$$

is a quasilinear equation.

3.3 Fully Nonlinear Equations

If the equation is nonlinear in u_x or u_y, then it is called fully nonlinear. Any equation that is not linear, semilinear, or quasilinear falls into this category.

4 Geometric Construction of First-Order PDEs

4.1 Family of Surfaces

Consider a family of surfaces given by

$$F(x,y,z,a,b) = 0,$$

where a and b are parameters and F is a C^1 function.

Assume that $z = z(x, y)$ is a function of x and y. Define

$$p = z_x, \quad q = z_y.$$

4.2 Differentiation

Differentiating with respect to x gives

$$F_x + pF_z = 0.$$

Differentiating with respect to y gives

$$F_y + qF_z = 0.$$

Together with

$$F(x, y, z, a, b) = 0,$$

we obtain three equations involving x, y, z, p, q, a, b.

Eliminating the parameters a and b yields a first-order PDE of the form

$$\Phi(x, y, z, p, q) = 0.$$

5 Complete Integral

If a first-order PDE

$$\Phi(x, y, z, p, q) = 0$$

can be solved to obtain a solution containing two arbitrary parameters, then this solution is called the *complete integral* of the PDE.

6 Example: Family of Spheres

Consider the family of spheres

$$x^2 + y^2 + (z - c)^2 = r^2,$$

where c and r are parameters.

Differentiating with respect to x and y gives

$$x + p(z - c) = 0,$$

$$y + q(z - c) = 0.$$

Eliminating $z - c$ leads to the first-order PDE

$$qx - py = 0.$$

7 Exercise

Find the first-order PDE corresponding to the family of surfaces

$$(x-a)^2 + (y-b)^2 + z^2 = r^2,$$

where a and b are parameters.

8 A Theorem on First-Order PDEs

Theorem

Let $\phi(x, y, z)$ and $\psi(x, y, z)$ be two functions, and suppose

$$F(\phi, \psi) = 0$$

for some smooth function F.

Then $z = z(x, y)$ satisfies the first-order PDE

$$\begin{vmatrix} \phi_x + p\phi_z & \psi_x + p\psi_z \\ \phi_y + q\phi_z & \psi_y + q\psi_z \end{vmatrix} = 0,$$

where $p = z_x$ and $q = z_y$.

Proof Sketch

Define

$$G(x, y, z) = F(\phi(x, y, z), \psi(x, y, z)).$$

Then $G = 0$ implies

$$G_x = 0, \quad G_y = 0.$$

Using the chain rule,

$$G_x = F_\phi(\phi_x + p\phi_z) + F_\psi(\psi_x + p\psi_z),$$

$$G_y = F_\phi(\phi_y + q\phi_z) + F_\psi(\psi_y + q\psi_z).$$

For a nontrivial solution (F_ϕ, F_ψ), the determinant of the coefficient matrix must vanish, which yields the stated PDE.

9 Conclusion

Families of surfaces naturally generate first-order PDEs. Conversely, solving a first-order PDE leads to a family of surfaces, whose complete integral contains two arbitrary parameters.