

Lecture 06: Introduction to Partial Differential Equations

1 Introduction

In this lecture, we introduce the concept of *partial differential equations (PDEs)*. We begin by recalling the definition of an *ordinary differential equation (ODE)* and then motivate the definition of a PDE as a natural generalization.

2 Ordinary Differential Equations (ODEs)

Let $I_1 = (a, b) \subset \mathbb{R}$ be an interval and let

$$u : I_1 \rightarrow \mathbb{R}$$

be an unknown function.

Consider a relation of the form

$$f(x, u(x), u'(x)) = 0, \tag{1}$$

where

$$f : I_1 \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$$

is a given smooth function.

2.1 First-Order ODE

Equation (1) is called a *first-order ordinary differential equation* if

$$\frac{\partial f}{\partial z}(x, y, z) \neq 0,$$

where we identify

$$y = u(x), \quad z = u'(x).$$

This condition ensures that the equation genuinely involves the derivative $u'(x)$.

2.2 Examples

- $f(x, y, z) = x$: Not an ODE (no derivative term).
- $f(x, y, z) = xy$: Not an ODE.
- $f(x, y, z) = z + xy$: A first-order ODE.

The equation

$$z + xy = 0$$

corresponds to

$$u'(x) + xu(x) = 0.$$

2.3 Higher-Order ODEs

A second-order ODE has the form

$$f(x, u, u', u'') = 0,$$

with

$$\frac{\partial f}{\partial w} \neq 0,$$

where $w = u''(x)$.

Example:

$$u'' + 1 = 0$$

is a second-order ODE.

3 Partial Differential Equations (PDEs)

We now generalize ODEs to PDEs.

Let

$$I = (a, b) \times (c, d) \subset \mathbb{R}^2$$

and let

$$u : I \rightarrow \mathbb{R}$$

be an unknown smooth function.

3.1 First-Order PDEs

Consider a relation

$$f(x, y, u, u_x, u_y) = 0, \tag{2}$$

where

$$f : I \times \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}$$

is a given smooth function.

Equation (2) is called a *first-order PDE* if at least one of

$$\frac{\partial f}{\partial p}, \quad \frac{\partial f}{\partial q}$$

is nonzero, where

$$p = u_x, \quad q = u_y.$$

3.2 Examples

- $f(x, y, z, p, q) = x + yz$: Not a PDE.
- $f(x, y, z, p, q) = p + q$: A first-order PDE.

The equation

$$p + q = 0$$

corresponds to

$$u_x + u_y = 0,$$

which is called a *transport equation*.

Burgers-Type Equation:

$$uu_x + u_y = 0.$$

4 Second-Order PDEs

If $u \in C^2(I)$, then the second-order derivatives are

$$u_{xx}, \quad u_{xy}, \quad u_{yy}.$$

A second-order PDE has the form

$$f(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0,$$

where at least one derivative with respect to u_{xx}, u_{xy}, u_{yy} is nonzero.

4.1 Important Examples

1. **Laplace (Harmonic) Equation:**

$$u_{xx} + u_{yy} = 0.$$

2. **Heat Equation:**

$$u_{xx} - u_y = 0.$$

3. **Wave Equation:**

$$u_{xx} - u_{yy} = 0.$$

5 Classical Solutions

A function u is called a *classical solution* of a PDE if:

- All derivatives appearing in the equation exist,
- The equation is satisfied for all $(x, y) \in I$.

Typically, we assume $u \in C^1$ for first-order PDEs and $u \in C^2$ for second-order PDEs.

5.1 Example

Consider

$$u_x = 0.$$

Then the general solution is

$$u(x, y) = \phi(y) + k,$$

where $\phi \in C^1$ and $k \in \mathbb{R}$.

6 General and Particular Solutions

- A *general solution* represents all possible solutions.
- A *particular solution* is obtained by imposing additional data (initial or boundary conditions).

7 Well-Posedness of a PDE

A PDE is called *well-posed* if:

1. A solution exists,
2. The solution is unique,
3. The solution depends continuously on the data.

8 Linear and Nonlinear PDEs

8.1 Linear PDE

A first-order PDE is linear if it has the form

$$\alpha(x, y)u_x + \beta(x, y)u_y + \gamma(x, y)u + \delta(x, y) = 0,$$

where $\alpha, \beta, \gamma, \delta$ depend only on x, y .

Example:

$$u_x + u_y = 0.$$

8.2 Nonlinear PDE

If the equation involves nonlinear terms such as uu_x, u_x^2 , etc., it is nonlinear.

Examples:

$$uu_x + u_y = 0,$$

$$u_x^2 + u_y^2 = 0.$$

9 Exercise

- Give three examples of **second-order linear PDEs**.
- Give three examples of **second-order nonlinear PDEs**.