Lecture 05 : Part 2
Basics of Calculus II

1 Functions of Several Variables

Let

$$u : I \subset \mathbb{R}^2 \to \mathbb{R},$$

where

$$I = (a, b) \times (c, d)$$

is an open rectangle. The function $u(x, y)$ is a *real-valued function of two variables.* At this stage, we do *not* assume continuity.

2 Motivation for Partial Derivatives

Fix $y \in (c, d)$. Define a new function

$$w_y(x) = u(x, y), \quad x \in (a, b).$$

Then $w_y : (a, b) \to \mathbb{R}$ is a one-dimensional function.

If w_y is differentiable at x, we define

$$w_y'(x) = \lim_{h \to 0} \frac{w_y(x + h) - w_y(x)}{h} = \lim_{h \to 0} \frac{u(x + h, y) - u(x, y)}{h}.$$

2.1 Partial Derivative with Respect to x

If the above limit exists, we define the *partial derivative of u with respect to x* at (x, y) as

$$\frac{\partial u}{\partial x}(x, y) = \lim_{h \to 0} \frac{u(x + h, y) - u(x, y)}{h}.$$

2.2 Partial Derivative with Respect to y

Similarly, fixing x and allowing y to vary, we define

$$\frac{\partial u}{\partial y}(x_0, y_0) = \lim_{k \to 0} \frac{u(x_0, y_0 + k) - u(x_0, y_0)}{k},$$

provided the limit exists.

3 Example: Existence of Partial Derivatives Without Continuity

Define

$$u(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 0, \\ 0, & \text{otherwise.} \end{cases}$$

3.1 Continuity

This function is *not continuous* at any point on the coordinate axes. For example, at $(x_0, 0)$, take a sequence (x_0, y_n) with $y_n \neq 0$ and $y_n \to 0$. Then

$$u(x_0, y_n) = 0 \not\to 1 = u(x_0, 0).$$

3.2 Partial Derivatives at the Origin

Consider

$$\frac{\partial u}{\partial x}(0,0) = \lim_{h \to 0} \frac{u(h,0) - u(0,0)}{h} = \lim_{h \to 0} \frac{1-1}{h} = 0.$$

Similarly,

$$\frac{\partial u}{\partial y}(0,0) = 0.$$

Conclusion. Partial derivatives may exist even when the function is not continuous. Unlike the one-dimensional case, existence of partial derivatives does *not* imply continuity.

4 Gradient and First Derivative

The *gradient* of u at (x,y) is defined as

$$\nabla u(x,y) = \left(\frac{\partial u}{\partial x}(x,y), \frac{\partial u}{\partial y}(x,y) \right).$$

We interpret the gradient as the *derivative* of u.

5 Continuously Differentiable Functions

A function u is said to be *continuously differentiable* (or C^1) on I if

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}$$

exist on I and are continuous.

6 Example

Let

$$u(x,y) = e^x \sin y.$$

Then

$$\frac{\partial u}{\partial x} = e^x \sin y, \quad \frac{\partial u}{\partial y} = e^x \cos y.$$

At $(1,0)$,

$$\nabla u(1,0) = (0, e).$$

7 Second-Order Derivatives and the Hessian

If $u \in C^1(I)$ and the gradient ∇u is differentiable, we define the *second derivative* as

$$D^2 u = \begin{pmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{pmatrix},$$

called the *Hessian matrix* of u.

A function u is called C^2 if all second-order partial derivatives exist and are continuous.

8 Example: Hessian Calculation

Let

$$u(x, y) = 4x + 3y^2.$$

Then

$$u_x = 4, \quad u_y = 6y,$$

$$u_{xx} = 0, \quad u_{xy} = 0, \quad u_{yx} = 0, \quad u_{yy} = 6.$$

Thus,

$$D^2 u = \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}.$$

9 The Laplacian

The *Laplacian* of u is defined as

$$\Delta u = u_{xx} + u_{yy}.$$

Example. Let

$$u(x, y) = x^2 - y^2.$$

Then

$$u_{xx} = 2, \quad u_{yy} = -2,$$

so

$$\Delta u = 0.$$

Such functions are called *harmonic functions*.

10 Directional Derivative

Let $\eta = (\eta_1, \eta_2)$ be a unit vector.

The *directional derivative* of u at (x_0, y_0) in the direction η is

$$\frac{\partial u}{\partial \eta}(x_0, y_0) = \lim_{t \to 0} \frac{u(x_0 + t\eta_1, y_0 + t\eta_2) - u(x_0, y_0)}{t},$$

provided the limit exists.

10.1 Relation with the Gradient

If $u \in C^1(I)$, then

$$\frac{\partial u}{\partial \eta}(x_0, y_0) = \nabla u(x_0, y_0) \cdot \eta.$$

11 Final Example

Let

$$u(x, y) = e^x + y^2.$$

Compute the directional derivative at $(1, 1)$ in the direction $\eta = (3, 4)$.
First normalize:

$$\hat{\eta} = \frac{1}{5}(3, 4).$$

Then

$$\nabla u(x, y) = (e^x, 2y),$$

$$\nabla u(1, 1) = (e, 2).$$

Hence,

$$\frac{\partial u}{\partial \eta}(1, 1) = (e, 2) \cdot \frac{1}{5}(3, 4) = \frac{3e + 8}{5}.$$