

# Lecture 05 : Part 1 – Basics of Calculus

## 1 Introduction and Motivation

In the study of Partial Differential Equations (PDEs), we are typically interested in finding a real-valued function

$$u : D \rightarrow \mathbb{R},$$

where  $D$  is a domain in  $\mathbb{R}^n$ . For simplicity and clarity, throughout this lecture we restrict our discussion to the case  $n = 2$ , i.e.  $D \subset \mathbb{R}^2$ . All the concepts discussed here extend naturally to higher dimensions.

## 2 Domains in $\mathbb{R}^2$

### 2.1 Domain

Unless otherwise specified, by a *domain*  $D$  we mean a set that is

- open, and
- connected.

### 2.2 Open Sets

A set  $D \subset \mathbb{R}^2$  is called *open* if for every point  $x \in D$ , there exists  $\varepsilon > 0$  such that the open ball

$$B(x, \varepsilon) = \{y \in \mathbb{R}^2 : \|y - x\| < \varepsilon\}$$

is completely contained in  $D$ .

#### Example: Unit Ball

The open unit ball in  $\mathbb{R}^2$  is defined as

$$B(0, 1) = \{x \in \mathbb{R}^2 : \|x\| < 1\}.$$

Its closure is

$$\overline{B(0,1)} = \{x \in \mathbb{R}^2 : \|x\| \leq 1\},$$

and its boundary is

$$\partial B(0,1) = \{x \in \mathbb{R}^2 : \|x\| = 1\}.$$

More generally, for  $x_0 \in \mathbb{R}^2$  and  $r > 0$ ,

$$B(x_0, r) = \{x \in \mathbb{R}^2 : \|x - x_0\| < r\}, \quad \partial B(x_0, r) = \{x \in \mathbb{R}^2 : \|x - x_0\| = r\}.$$

## 2.3 Connected Sets

A set  $D \subset \mathbb{R}^2$  is said to be *connected* if for any two points in  $D$ , there exists a continuous path joining them that lies entirely in  $D$ .

Intuitively, this means that the set has no breaks or separations.

## 2.4 Rectangular Domains

In many PDE problems, we work with rectangular domains of the form

$$I = (a, b) \times (c, d),$$

which is an open and connected subset of  $\mathbb{R}^2$ .

Its closure is

$$\bar{I} = [a, b] \times [c, d],$$

which includes the boundary of the rectangle.

# 3 Functions on Domains

Let

$$u : I \rightarrow \mathbb{R}$$

be a real-valued function defined on a domain  $I \subset \mathbb{R}^2$ .

# 4 Continuity of Functions of Two Variables

## 4.1 Definition (Sequential Form)

A function  $u : I \rightarrow \mathbb{R}$  is said to be *continuous at a point*  $(x_0, y_0) \in I$  if for every sequence  $(x_n, y_n) \subset I$  such that

$$(x_n, y_n) \rightarrow (x_0, y_0),$$

we have

$$u(x_n, y_n) \rightarrow u(x_0, y_0).$$

If  $u$  is continuous at every point of  $I$ , then  $u$  is said to be continuous on  $I$ .

## 4.2 Interpretation

Continuity means that the value of the function at  $(x_0, y_0)$  does not depend on the path along which we approach the point. Along any path approaching  $(x_0, y_0)$ , the function values must approach  $u(x_0, y_0)$ .

# 5 Examples of Continuous Functions

## 5.1 Constant Function

Let

$$u(x, y) = k, \quad k \in \mathbb{R}.$$

Then for any sequence  $(x_n, y_n)$ ,

$$u(x_n, y_n) = k \rightarrow k = u(x_0, y_0).$$

Hence, constant functions are continuous everywhere.

## 5.2 Product of Continuous Functions

Let

$$u(x, y) = \phi(x)\psi(y),$$

where  $\phi$  is continuous on  $(a, b)$  and  $\psi$  is continuous on  $(c, d)$ . Then  $u$  is continuous on  $I$

### Sketch of Proof

Let  $(x_n, y_n) \rightarrow (x_0, y_0)$ . Then

$$\begin{aligned} |u(x_n, y_n) - u(x_0, y_0)| &= |\phi(x_n)\psi(y_n) - \phi(x_0)\psi(y_0)| \\ &= |\phi(x_n)(\psi(y_n) - \psi(y_0)) + (\phi(x_n) - \phi(x_0))\psi(y_0)| \\ &\leq |\phi(x_n)| |\psi(y_n) - \psi(y_0)| + |\psi(y_0)| |\phi(x_n) - \phi(x_0)|. \end{aligned}$$

Since convergent sequences are bounded and both  $\phi$  and  $\psi$  are continuous, the right-hand side tends to zero. Hence  $u$  is continuous.

### 5.3 Trigonometric Example

Let

$$u(x, y) = \sin x \cos y.$$

Since  $\sin x$  and  $\cos y$  are continuous, their product is continuous on  $\mathbb{R}^2$ .

## 6 Linear Combinations of Continuous Functions

Let

$$u(x, y) = a \phi(x) + b \psi(y),$$

where  $a, b \in \mathbb{R}$  and  $\phi, \psi$  are continuous functions.

Then  $u$  is continuous, since sums and scalar multiples of continuous functions are continuous.

### Exercise

Prove the above result using the definition of continuity.

## 7 Conclusion

Continuity is a fundamental concept in PDE theory. Many functions encountered in PDEs are constructed using sums, products, and compositions of simpler continuous functions. A solid understanding of continuity in  $\mathbb{R}^2$  forms the foundation for studying differentiability and higher-order properties.