Lecture 04: Part C
Behavior of Regular Sturm–Liouville Problems (Part III)

1 Regular Sturm–Liouville Boundary Value Problems

We consider a *regular Sturm–Liouville boundary value problem* of the form

$$L[y] + \lambda s(x) y = 0, \qquad x \in (a, b), \tag{1}$$

where the Sturm–Liouville operator L is defined by

$$L[y] := \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y.$$

The boundary conditions are

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \tag{2}$$

$$\beta_1 y(b) + \beta_2 y'(b) = 0, \tag{3}$$

with

$$\alpha_1^2 + \alpha_2^2 \neq 0, \qquad \beta_1^2 + \beta_2^2 \neq 0.$$

These conditions ensure that we are dealing with a genuine boundary value problem.

Eigenvalue Problem

The goal is to find all values $\lambda \in \mathbb{R}$ for which there exists a *nontrivial* solution $y \not\equiv 0$ of (1) satisfying the boundary conditions. Such values of λ are called *eigenvalues*, and the corresponding solutions are called *eigenfunctions*.

2 Example: A Simple Sturm–Liouville Problem

Consider the problem

$$y'' + \lambda y = 0, \qquad x \in (0, 1), \tag{4}$$

with boundary conditions

$$y(0) = 0, \tag{5}$$

$$y(1) + y'(1) = 0. \tag{6}$$

This is a regular Sturm–Liouville problem with

$$p(x) = 1, \quad q(x) = 0, \quad s(x) = 1.$$

General Solution

The general solution of (4) is

$$y(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

Applying the condition $y(0) = 0$ gives

$$A = 0,$$

so that

$$y(x) = B\sin(\sqrt{\lambda}x).$$

Eigenvalue Equation

We now apply the second boundary condition:

$$y(1) + y'(1) = 0.$$

Since

$$y'(x) = B\sqrt{\lambda}\cos(\sqrt{\lambda}x),$$

we obtain

$$B\sin(\sqrt{\lambda}) + B\sqrt{\lambda}\cos(\sqrt{\lambda}) = 0.$$

For a nontrivial solution $B \neq 0$, this yields the *eigenvalue equation*

$$\tan(\sqrt{\lambda}) = -\sqrt{\lambda}. \tag{7}$$

Existence of Eigenvalues

Equation (7) cannot be solved explicitly. However, by studying the graphs of

$$y = \tan x \quad \text{and} \quad y = -x,$$

one observes that there is at least one root of

$$h(x) = \tan x + x$$

in each interval

$$\left(\frac{\pi}{2} + k\pi,\ \frac{3\pi}{2} + k\pi\right), \qquad k = 0, 1, 2, \ldots$$

By the Intermediate Value Theorem, each such interval contains exactly one root α_n. Hence, the eigenvalues are

$$\lambda_n = \alpha_n^2, \qquad n = 1, 2, 3, \ldots$$

and satisfy

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots, \qquad \lim_{n\to\infty} \lambda_n = \infty.$$

Eigenfunctions

The corresponding eigenfunctions are

$$\phi_n(x) = \sin(\sqrt{\lambda_n}\, x),$$

unique up to a constant multiple.

3 General Theorem for Regular Sturm–Liouville Problems

[Spectral Theorem for Regular Sturm–Liouville Problems] A regular Sturm–Liouville boundary value problem has:

1. An infinite sequence of real eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots,$$

2. Eigenfunctions $\{\phi_n\}$ that are unique up to multiplication by constants,
3. Orthogonality of eigenfunctions with respect to the weight function $s(x)$:

$$\int_a^b s(x)\phi_m(x)\phi_n(x)\, dx = 0, \quad m \neq n,$$

4. Exactly n zeros of ϕ_n in the interval (a, b).

The proof of this theorem is nontrivial and can be found in classical texts such as *Coddington and Levinson, Theory of Ordinary Differential Equations*.

4 Periodic Sturm–Liouville Problems

A *periodic Sturm–Liouville problem* has the same differential equation

$$L[y] + \lambda s(x)y = 0,$$

but with periodic boundary conditions

$$y(a) = y(b), \tag{8}$$
$$y'(a) = y'(b), \tag{9}$$

together with the assumption

$$p(a) = p(b).$$

Properties

For periodic Sturm–Liouville problems:

- Eigenvalues are real,
- Eigenfunctions corresponding to distinct eigenvalues are orthogonal,
- The operator remains self-adjoint.

Important Difference

However, the nodal property (exactly n zeros for the nth eigenfunction) *does not hold* for periodic Sturm–Liouville problems. This marks a fundamental difference between regular and periodic Sturm–Liouville systems.

5 Conclusion

Regular Sturm–Liouville problems possess a rich and well-ordered spectral structure, including:

- Existence of infinitely many eigenvalues,
- Orthogonality and completeness of eigenfunctions,
- Precise control over nodal behavior.

While many of these properties extend to periodic Sturm–Liouville problems, some key results-particularly nodal theorems-fail in the periodic case.