

Lecture 01 - Part A

Periodic Functions

1 Introduction

In this lecture, we begin the study of Fourier analysis by introducing *periodic functions*. Periodic functions play a fundamental role in Fourier analysis, since Fourier series aim to represent functions using sine and cosine functions, which are themselves periodic.

2 Definition of a Periodic Function

Let

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

be a real-valued function.

A function $f(x)$ is called *periodic* if:

- $f(x)$ is defined for all $x \in \mathbb{R}$ (except possibly at a finite number of points), and
- there exists a real number $p > 0$ such that

$$f(x + p) = f(x) \quad \text{for all } x \in \mathbb{R}.$$

The number p is called a *period* of the function f .

This condition means that the function repeats its values after every interval of length p .

3 Examples of Periodic Functions

3.1 Trigonometric Functions

Consider

$$f(x) = \sin x \quad \text{or} \quad f(x) = \cos x.$$

Since

$$\sin(x + 2\pi) = \sin x, \quad \cos(x + 2\pi) = \cos x$$

for all $x \in \mathbb{R}$, both $\sin x$ and $\cos x$ are periodic functions with period 2π .

4 Non-Uniqueness of Periods

A periodic function may have more than one period. For example, if 2π is a period of $\sin x$, then $4\pi, 6\pi, \dots$ are also periods.

The smallest positive period of a periodic function is called its *fundamental period*.

For $\sin x$ and $\cos x$, the fundamental period is 2π .

5 A Basic Property of Periods

If p is a period of a function f , then np is also a period of f for every $n \in \mathbb{N}$.

Proof. For $n = 2$,

$$f(x + 2p) = f((x + p) + p) = f(x + p) = f(x),$$

since p is a period of f . The argument extends similarly to any $n \in \mathbb{N}$. □

6 Periodic Functions Not Defined Everywhere

A function need not be defined at every point of \mathbb{R} in order to be periodic.

6.1 Example: The Tangent Function

The function

$$f(x) = \tan x$$

is periodic with period π , although it is not defined at

$$x = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}.$$

7 Non-Periodic Functions

7.1 Polynomial Functions

Let

$$f(x) = x^n, \quad n \in \mathbb{N}.$$

If f were periodic with period $p > 0$, then

$$(x + p)^n = x^n \quad \text{for all } x \in \mathbb{R},$$

which is impossible. Hence polynomial functions are not periodic.

7.2 The Exponential Function

Consider

$$f(x) = e^x.$$

If e^x were periodic with period $p > 0$, then

$$e^{x+p} = e^x \quad \text{for all } x \in \mathbb{R}.$$

This implies $e^p = 1$, so $p = 0$, which contradicts the definition of a period. Therefore, e^x is not periodic.

8 Algebraic Properties of Periodic Functions

Let $a, b \in \mathbb{R}$ and let f and g be p -periodic functions. Then the function

$$h(x) = af(x) + bg(x)$$

is also p -periodic.

Proof. Since $f(x + p) = f(x)$ and $g(x + p) = g(x)$,

$$h(x + p) = af(x + p) + bg(x + p) = af(x) + bg(x) = h(x).$$

□

9 Why Study Periodic Functions?

The goal of Fourier analysis is to represent functions using periodic functions such as

$$\sin(nx), \quad \cos(nx), \quad n \in \mathbb{N}.$$

Understanding periodic functions is therefore the first step toward studying Fourier series.

10 Conclusion

Periodic functions form the foundation of Fourier analysis. In the next lecture, we will study how functions can be represented as sums of sine and cosine functions using Fourier series.