Optimization Algorithms: Theory and Software Implementation

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Lecture: 4

Hello, everyone. So, this is the fourth lecture in the course. Recall that in the previous lecture, we were trying to minimize a function: $f(x_1,x_2,\ldots,x_n)$, where f is a function of n independent variables. Each x_i can take any value in R.

To find the minimum, we first find the set of critical points, i.e., points where the gradient of f is zero: $\nabla f = 0$

For each critical point $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$, we compute the Hessian matrix $\nabla^2 f(x^*)$. We have the following characterizations:

If $t^\top \nabla^2 f(x^*) t > 0$ for all $t \neq 0$, then x^* is a local minimum.

If $t^\top \nabla^2 f(x^*) t < 0$ for all $t \neq 0$, then x^* is a local maximum.

If there exists t_1, t_2 such that $t_1^\top \nabla^2 f(x^*) t_1 > 0$ and $t_2^\top \nabla^2 f(x^*) t_2 < 0$, then x^* is a saddle point.

If $t^\top \nabla^2 f(x^*) t = 0$ for some $t \neq 0$, further investigation is needed.

Examples

1. Let $n = 2$, and $f(x_1, x_2) = x_1^2 + x_2^2$.

$$\nabla f = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} \Rightarrow \nabla f = 0 \text{ at } x_1 = 0, x_2 = 0$$

Hessian:

$$\nabla^2 f(x^*) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

For any $t = (t_1,t_2)$, $t^\top \nabla^2 f(x^*) t = 2t_1^2 + 2t_2^2 > 0$ for all $t \neq 0$. So, $x^* = (0,0)$ is a **local minimum**.

2. $f(x_1, x_2) = -x_1^2 - x_2^2$

$$\nabla f = \begin{bmatrix} -2x_1 \\ -2x_2 \end{bmatrix} \Rightarrow x^* = (0,0)$$

$$\nabla^2 f(x^*) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$t^\top \nabla^2 f(x^*) t = -2t_1^2 - 2t_2^2 < 0 \text{ for all } t \neq 0, \textbf{ Local maximum.}$$

3. $f(x_1, x_2) = x_1^2 - x_2^2$

$$\nabla f = \begin{bmatrix} 2x_1 \\ -2x_2 \end{bmatrix} \Rightarrow x^* = (0,0)$$

$$\nabla^2 f(x^*) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$t^\top \nabla^2 f(x^*) t = 2t_1^2 - 2t_2^2 \quad ,$$

Saddle point since this quantity can be positive or negative.

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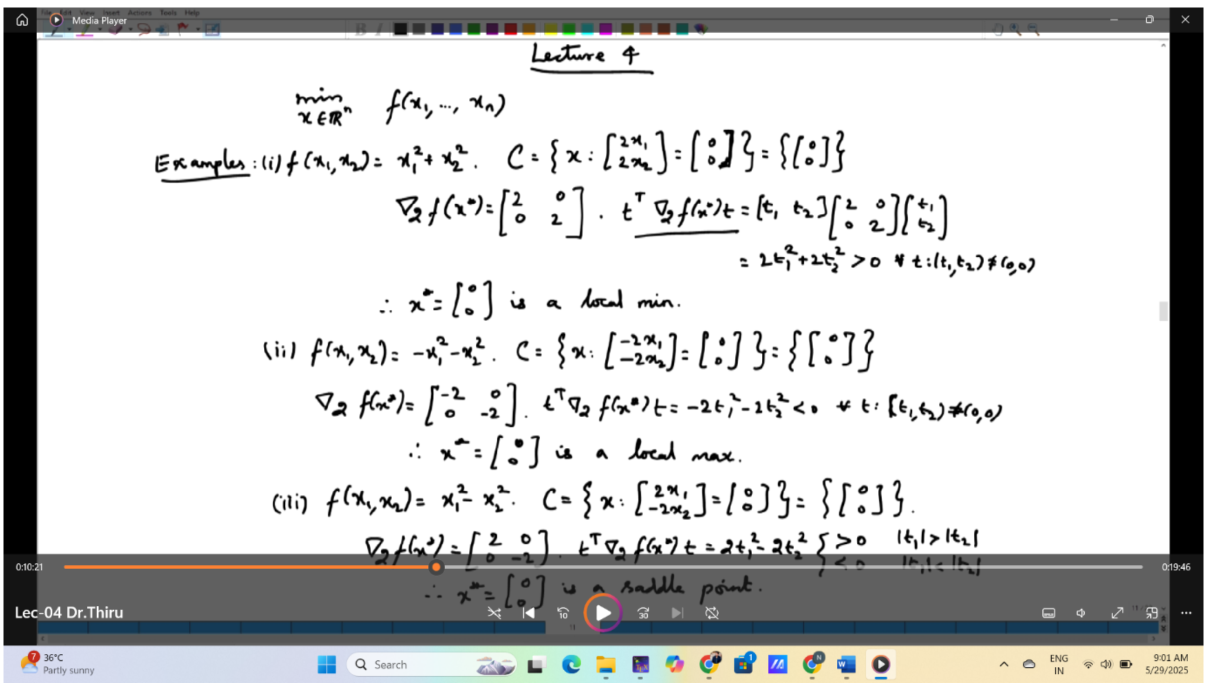

4. $f(x_1, x_2) = x_1^2 - x_2^4$

$$\nabla f = \begin{bmatrix} 2x_1 \\ -4x_2^3 \end{bmatrix} \Rightarrow x^* = (0,0)$$

$$\nabla^2 f(x^*) = \begin{bmatrix} 2 & 0 \\ 0 & -8x_2^2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$t^\top \nabla^2 f(x^*) t = 2t_1^2$$

This is ≥ 0, but equals 0 when $t_1 = 0$, $t_2 \neq 0$. So, **further probe is needed**.

Consider:

$$f(0,\epsilon) = -\epsilon^4 < 0, \qquad f(\epsilon,0) = \epsilon^2 > 0, f(0,0) = 0, \textbf{ Saddle point.}$$

5. $f(x_1, x_2) = x_1^2 + x_2^4$

$$\nabla f = \begin{bmatrix} 2x_1 \\ 4x_2^3 \end{bmatrix} \Rightarrow x^* = (0,0)$$

$$\nabla^2 f(x^*) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$t^\top \nabla^2 f(x^*) t = 2t_1^2 \geq 0$$

Further probe needed. But since $f(x_1,x_2) \geq 0 = f(0,0)$, **local and global minimum**.

6. $f(x_1, x_2) = -x_1^2 - x_2^4$, Same analysis:

$$\nabla^2 f(x^*) = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}, \quad t^\top \nabla^2 f(x^*) t = -2t_1^2 \leq 0$$

Further probe needed, but since $f(x_1,x_2) \leq 0 = f(0,0)$, **local maximum**.

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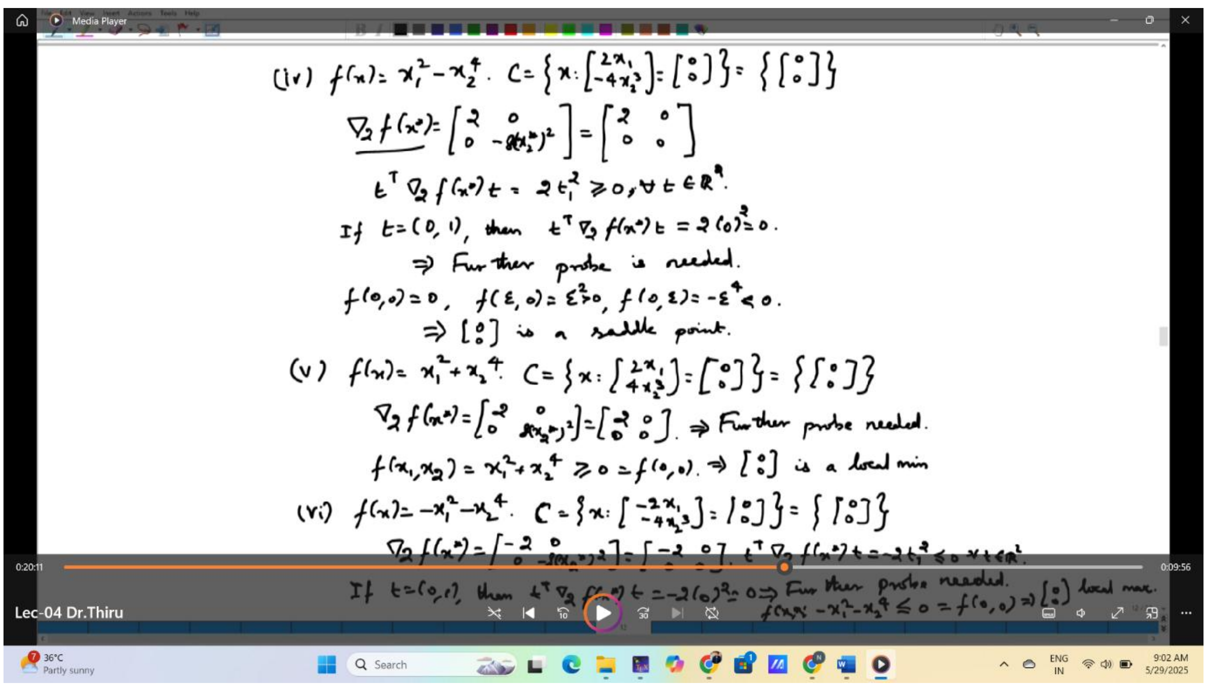

Summary: Unconstrained Optimization

To find local solutions:

1. Find critical points: $\nabla f = 0$ 2.

Evaluate the Hessian $\nabla^2 f(x^*)$

3. Analyze $t^{\top}\nabla^2 f(x^*)t$:

$> 0 \Rightarrow$ local min

$< 0 \Rightarrow$ local max

< 0 and $> 0 \Rightarrow$ saddle point

$= 0$ for some $t \Rightarrow$ further analysis required

Constrained Optimization

We now consider:

$$\mathrm{Min} f(x)\ x \in S$$

If $S \neq \mathrm{R}^n$, this is a **constrained optimization**.

For example, if $S = \mathrm{Z}$, standard calculus-based methods do not work. We consider:

Min $f(x)$ subject to $g_i(x) \leq 0$ for $i = 1,...,p$ and $h_j(x) = 0$ for $j = 1,...,m$

Example from Lecture 1: Consumer Utility Maximization

Max $u(x_1,...,x_n)$ subject to

$$p_1x_1 + \cdots + p_nx_n \leq w,\ x_i \geq 0 \text{ for all } i$$

We rewrite this as: $\min -u(x_1,...,x_n)$.

Subject to:

$$p_1x_1 + \cdots + p_nx_n - w \leq 0$$
$$-x_1 \leq 0$$
$$...$$
$$-x_n \leq 0$$

So, now you can see that this problem is actually equivalent to what you have written here, where $f(x)$.

So, in this problem, f is actually $-u$, and $g_1 = p_1x_1 + \cdots + p_nx_n - w$, so that is less than or equal to 0, and

$$g_2 = -x_1, \qquad g_3 = -x_2, \qquad \ldots, \qquad g_{n+1} = -x_n.$$

So, this is a problem with $p = n+1$ and $m = 0$, because you do not have any equality constraints. So yeah, this actually defines—as you all know—it actually defines a constraint set.

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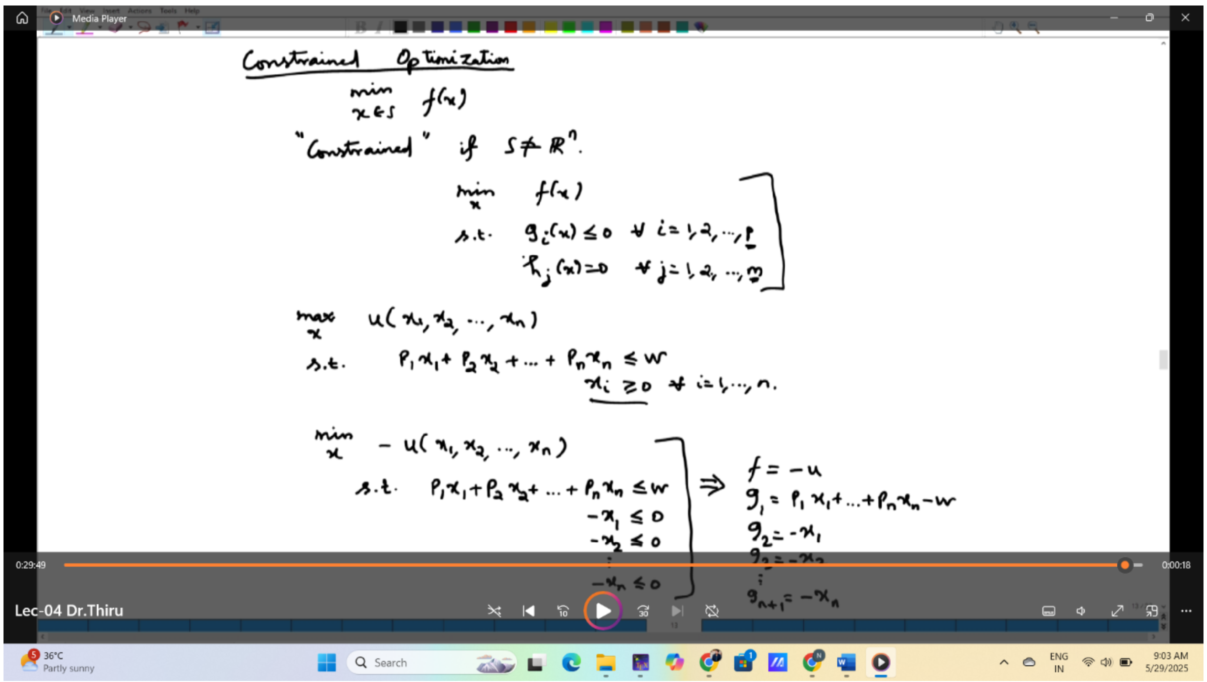

In the next lecture, we will learn how to define a constraint set and find the solution of a constrained optimization problem when it is of this particular form:

Min $f(x)$

subject to $g_i(x) \leq 0$ for $i = 1\ldots, p,$ and $h_j(x) = 0$for $j = 1\ldots, m.$

Thankyou