

An Introduction to Hyperbolic Geometry

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Module - 03

Lecture - 09

Introduction to the Disk Model of the Hyperbolic Plane

Hello! Today, we will discuss the upper half-plane model of the hyperbolic plane, which is one of several models available for this fascinating geometry. In this lecture, we will introduce the disk model of the hyperbolic plane. Importantly, this disk model is isometric to the upper half-plane when we consider the hyperbolic metric.

To explore this model, we will focus on the unit disk and define a hyperbolic metric on that disk. This will allow us to analyze the properties of the hyperbolic plane through the lens of the disk model.

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Disk Model of Hyperbolic Plane

Consider the Möbius $T: \mathbb{C} \cup \{\infty\} \rightarrow \mathbb{C} \cup \{\infty\}$

$$T(z) := \frac{z-i}{z+i}$$

Let $z = x \in \mathbb{R}$. Then $|T(x)| = \left| \frac{x-i}{x+i} \right| = 1$

$\Rightarrow T(x)$ lies in a unit circle S^1 .

$T(\infty) = 1 \in S^1$

Let $e^{i\theta} \in S^1$ & let $x = \frac{i(1+e^{i\theta})}{1-e^{i\theta}}$.

Then $T(x) = e^{i\theta}$ and $x \in \mathbb{R} \cup \{\infty\}$.

$T(\mathbb{R} \cup \{\infty\}) = S^1$.

Let $z \in \mathbb{H}^2$ & let $z = x+iy$

$$|T(z)| = \left| \frac{z-i}{z+i} \right| = \sqrt{\frac{x^2 + (y-1)^2}{x^2 + (y+1)^2}}$$

$y > 0 \Rightarrow 4y > 0 \Rightarrow y^2 - 2y + 1 < y^2 + 2y + 1$

$\Rightarrow (y-1)^2 < (y+1)^2$

$\Rightarrow x^2 + (y-1)^2 < x^2 + (y+1)^2$

This model will indeed be isometric to the upper half-plane with the hyperbolic metric that we have established. So, let us begin our exploration of the hyperbolic plane model.

First, we need to consider the Möbius transformation. This is key to our approach to the disk model, so let's dive in. We will examine the Möbius transformation T defined from $\mathcal{C} \cup \{\infty\}$ to $\mathcal{C} \cup \{\infty\}$ by the equation

$$T(z) = \frac{z - i}{z + i}.$$

Now, if we take z to be a real number, we find that the modulus of $T(x)$, where x is real, can be computed as follows:

$$|T(x)| = \left| \frac{x - i}{x + i} \right| = 1.$$

This implies that $T(x)$ lies on the unit circle, which we denote as S^1 . It is also worth noting that $T(\infty) = 1$, and this point is indeed part of S^1 .

Next, let us take an element $e^{i\theta}$ in S^1 and set

$$X = \frac{i(1 + e^{i\theta})}{1 - e^{i\theta}}.$$

Now, evaluating $T(X)$ yields

$$T(X) = e^{i\theta}.$$

You can verify that this X belongs to $R \cup \{\infty\}$.

Thus, we can conclude that the image of this transformation, $T(R \cup \{\infty\})$, is indeed the entire unit circle.

Now, let's consider an element in the upper half-plane. Let's set $z = x + iy$, where $y > 0$. Applying the transformation gives us:

$$T(z) = \frac{z - i}{z + i} = \frac{(x + iy) - i}{(x + iy) + i} = \frac{x + i(y - 1)}{x + i(y + 1)}.$$

Now, if we calculate the modulus, we find:

$$|T(z)| = \frac{\sqrt{x^2 + (y-1)^2}}{\sqrt{x^2 + (y+1)^2}}$$

Given that $y > 0$, we can infer that $4y > 0$, leading us to the conclusion that:

$$y^2 - 2y + 1 < y^2 + 2y + 1.$$

This can be simplified to:

$$(y-1)^2 < (y+1)^2.$$

From this, we can further deduce that

$$x^2 + (y-1)^2 < x^2 + (y+1)^2.$$

This is the relationship we are aiming to establish in our exploration of the disk model of the hyperbolic plane.

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Therefore, $|T(z)| = \left| \frac{z-i}{z+i} \right| < 1$
 $T(z) \in \mathbb{D}^2 = \{ w \in \mathbb{C} \mid |w| < 1 \}$
 Thus, T maps \mathbb{H}^2 to \mathbb{D}^2 .
 Let $w = T(z) = \frac{z-i}{z+i}$
 $\frac{dw}{dz} = \frac{2i}{(z+i)^2}$
 $|dw| = \frac{2}{|z+i|^2} |dz|$
 $\frac{|dw|^2}{(1-|w|^2)^2} = \frac{1}{\left(1 - \left|\frac{z-i}{z+i}\right|^2\right)^2} \times \frac{4}{|z+i|^4} |dz|^2$
 $= \frac{4 |dz|^2}{(|z+i|^2 - |z-i|^2)^2}$
 $= \frac{|dz|^2}{(2-2y)^2} \quad (\text{check})$

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Therefore, the modulus of $T(z)$ can be expressed as

$$|T(z)| = \left| \frac{z-i}{z+i} \right| < 1.$$

This implies that $T(z)$ lies within the open unit disk. Thus, we can conclude that the mapping T effectively transforms the upper half-plane into this open unit disk.

Now, let's introduce a new variable. Let $w = T(z)$. If we take the derivative of w with respect to z , we find

$$\frac{dw}{dz} = \frac{2i}{(z+i)^2}.$$

If we take the modulus of both sides, we obtain:

$$\left| \frac{dw}{dz} \right| = \frac{2}{|z+i|^2} |dz|.$$

Next, we can express the relation for $|dw|^2$ in terms of $|w|$:

$$|dw|^2 = (1 - |w|^2) \Rightarrow |dw|^2 = \frac{1}{1 - \left| \frac{z-i}{z+i} \right|^2}.$$

Now, if we simplify this, we get:

$$|dw|^2 = \frac{1}{1 - \left| \frac{z-i}{z+i} \right|^2} = \frac{1}{1 - \frac{|z-i|^2}{|z+i|^2}}.$$

By manipulating this expression further, we derive:

$$|dw|^2 = \frac{|dz|^2}{|z+i|^4}.$$

Consequently, we can observe that:

$$|dw|^2 = \frac{4|dz|^2}{|z+i|^4}$$

Thus, we can conclude that

$$|dw|^2 = \frac{4|dz|^2}{(z - z^*)^2}$$

where z^* denotes the complex conjugate of z .

So now, we have established the necessary relationships for our transformation and can proceed with the implications of these calculations.

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$$\frac{4 dw d\bar{w}}{(1 - |w|^2)^2} = \frac{dz d\bar{z}}{(\text{Im}(z))^2} \quad (*)$$

$w \in \mathbb{D}^2, |w| < 1$
 $\mathbb{D}^2 = \{ w \in \mathbb{C} : |w| < 1 \}$. Equip \mathbb{D}^2 with the
 metric $ds^2 = \frac{4 dw d\bar{w}}{(1 - |w|^2)^2}$

(\mathbb{D}^2, ds^2) is called the Poincaré disk model of the hyperbolic plane.

From $(*)$, the map $T: \mathbb{H}^2 \rightarrow \mathbb{D}^2$ is an isometry. The map T is conformal i.e. it preserves angles.

$w = u + iv$
 $ds^2 = \frac{4 (du^2 + dv^2)}{(1 - (u^2 + v^2))^2}$

$z \mapsto \frac{z-i}{z+i}$

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So, we have the expression:

$$\frac{4 dw d\bar{w}}{(1 - |w|^2)^2} = \frac{dz d\bar{z}}{(\text{Im}(z))^2}$$

where $z - \bar{z}$ is equal to twice the imaginary part of z . Hence, we have derived this

important formula.

Now, it is essential to note that w belongs to the unit disk, which implies that the modulus of w is less than 1. Next, we will take this unit disk and equip it with a metric, which we will refer to as the Riemannian metric. You can also treat it simply as a formula for the metric ds^2 , akin to what we established in the upper half-plane model with its respective metric.

In fact, we can express this metric in terms of dx and dy as follows. If we write w as $u + iv$, then we find:

$$ds^2 = \left(\frac{4 du^2 + dv^2}{(1 - |w|^2)\sqrt{u^2 + v^2}} \right)^2.$$

This metric ds^2 is known as the disk model of the hyperbolic plane. Furthermore, from our earlier formula, we have demonstrated that the mapping T from the upper half-plane to the unit disk is indeed an isometry. This is significant because it means that the mapping preserves the line element and consequently the metric itself.

Additionally, we note that since T is a Möbius transformation, it has certain properties. Specifically, the transformation $T(z) = \frac{z-i}{z+i}$ is conformal, meaning it preserves angles.

Now, the next question that arises is: What are the geodesics in the disk model of this hyperbolic plane? We have already established that in the upper half-plane, with respect to the hyperbolic metric, the geodesics are represented by vertical lines and semicircles that are orthogonal to the real axis.

Now, let's highlight an important point: any Möbius transformation preserves the set of circles and straight lines. This fact is particularly relevant here because our transformation T is indeed a Möbius transformation, mapping from the upper half-plane to the unit disk. If we take any straight line in the upper half-plane, its image under this transformation will be either a circle or another straight line.

Let's define L as a geodesic in the upper half-plane. In this context, L can either be a

vertical line or a semicircle that is orthogonal to the real axis, R . Therefore, we can conclude that L is a component of either a vertical line or a semicircle that intersects the real axis orthogonally.

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- Any Möbius transformation keeps the set of circles and straight lines invariant.

Let L be a geodesic in \mathbb{H}^2 . Then L is a part of either a vertical line or a semi-circle orthogonal to \mathbb{R} . Then $T(L)$ is either a straight line or a circular arc. L intersects \mathbb{R} orthogonally. $T(\mathbb{R}) = S^1$. $T(L)$ intersects S^1 orthogonally.

$\mathbb{H}^2 \longrightarrow D^2$

$T(z) = \frac{z-i}{z+i}$

Geodesics in D^2 (Disc Model of hyperbolic plane) are part of diameters & circular arcs intersecting S^1 orthogonally.

Since T is a Möbius transformation, the image of L under T will also be represented as either a straight line or a circular arc. It's crucial to note that since L intersects R orthogonally, the transformation T applied to R will yield a circle. Consequently, the image $P(L)$ will intersect the unit circle S^1 orthogonally.

Now, let's visualize the upper half-plane and the associated geodesics. We can draw vertical lines representing geodesics, while labeling one of them as i and introducing another geodesic nearby. This gives us a clearer understanding of our setup. The mapping $T(z) = \frac{z-i}{z+i}$ transforms this upper half-plane into the unit disk.

Importantly, when we evaluate $T(i)$, we find that it equals zero. So, $T(i)$ corresponds to the origin in the unit disk. Suppose we designate L as the vertical line passing through i , which

intersects R orthogonally at the origin. Because the transformation T is conformal, it preserves angles, meaning that the angle of intersection will remain unchanged. Thus, the image of L under T must pass through the origin.

In this case, $T(L)$ will represent the diameter of the unit disk. For $T(0)$, we obtain -1 , and as $T(\infty)$ equals 1 , we can visualize $T(L)$ extending along this diameter.

Next, consider a red geodesic that passes through i and is shaped as a semicircle intersecting R orthogonally. Again, $T(L)$ will pass through the origin, leading us to conclude that it, too, represents a diameter. Thus, it takes the form of a straight line.

Now, regarding this green geodesic, which does not pass through i : the image T of this green geodesic will again be a geodesic but will not pass through the point i . Instead, it will manifest as a circular arc that intersects the circumference of the unit disk, specifically the unit circle S^1 , orthogonally.

Thus, the geodesics in the disk model of the hyperbolic plane consist of parts of diameters and circular arcs that intersect the unit circle orthogonally. These observations help us deepen our understanding of the geometric structure we are exploring, and one can verify this property with careful consideration of the transformations involved.

We have observed that the isometry group of the upper half-plane, specifically the orientation-preserving isometries, is precisely equal to $PSL(2, R)$. This is a critical point worth noting. Now, if we consider the orientation-preserving isometries of the Poincaré disk model, or this upper half-plane model, we find that they correspond to $PSU(1,1)$.

To clarify, $PSU(1,1)$ consists of matrices of the form

$$\begin{pmatrix} a & c \\ \bar{c} & a \end{pmatrix}$$

where a and c are complex numbers, and the determinant of this matrix equals 1 , modulo the subgroup containing \pm Identity.

Now, we know that there exists an isometry mapping from the upper half-plane to the unit

disk. This isometry can be utilized to demonstrate that $PSL(2, R)$ maps under our transformation T to $PSU(1, 1)$.

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- check that $\text{Isom}^+(\mathbb{D}^2) = \text{PSU}(1, 1)$
 where $\text{PSU}(1, 1) = \left\{ \begin{pmatrix} a & c \\ \bar{c} & a \end{pmatrix} : a, c \in \mathbb{C}, |a|^2 - |c|^2 = 1 \right\} / \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$
- $d_{\mathbb{D}^2}(z, w) = \ln \left(\frac{|1 - z\bar{w}| + |z - w|}{|1 - z\bar{w}| - |z - w|} \right)$ (Exercise)
 $z, w \in \mathbb{D}^2$

Unit disc model of hyperbolic plane is also called Poincaré Disc.

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In the upper half-plane, we also have a distance formula, which is applicable here as well. I will present this distance formula as an exercise for you to explore. It can be expressed as:

$$d(z, w) = \ln \left(\frac{|1 - z\bar{w}| + |z - w|}{|1 - z\bar{w}| - |z - w|} \right)$$

This relationship holds when z and w are both contained within the unit disk model, which is also referred to as the Poincaré disk model.

We will be utilizing both models of the hyperbolic plane, as they are isometric to each other. Therefore, whenever necessary, we can choose to work with either the upper half-plane model or the disk model, depending on the context or the problem at hand. Okay, I'll stop here and in the next lecture, we'll be doing hyperbolic trigonometry.