

An Introduction to Hyperbolic Geometry

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Module - 3

Lecture - 8

Proof of Hyperbolic Triangle Area Formula Using Cauchy-Riemann Equations and Isometries

Hello, and welcome to this lecture on hyperbolic geometry! In today's session, we will prove a fascinating result: if you take a geodesic triangle in the upper half-plane concerning the hyperbolic metric, then the area of that triangle will equal π minus the sum of its vertex angles.

To establish this theorem, we will need to consider some foundational concepts from complex analysis, specifically the fact that analytic functions satisfy the Cauchy-Riemann equations.

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Hyperbolic Area:- Let $A \subseteq \mathbb{H}^2$, \mathbb{H}^2 , $ds^2 = \frac{dx^2 + dy^2}{y^2}$
We define $\mu(A)$, the hyperbolic area of A , by
$$\mu(A) := \int_A \frac{dx dy}{y^2} \quad \text{if the integral exists}$$

Cauchy Riemann Equations:- An analytic function $f(z) = u(x,y) + i v(x,y)$
Where $z = x + iy \in \Omega$ (Domain) satisfies Cauchy-Riemann
Equations. i.e. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Theorem:- Let $A \subseteq \mathbb{H}^2$ be such that $\mu(A)$ exists
Let $T \in \text{PSL}(2, \mathbb{R})$. Then $\mu(T(A)) = \mu(A)$

Proof:- Let $T(z) = \frac{az + b}{cz + d}$, $ad - bc = 1$, $a, b, c, d \in \mathbb{R}$
Let $z = x + iy$ & $T(z) = u(x,y) + i v(x,y)$.

Let's begin by defining what we mean by hyperbolic area. Suppose we have a subset A of the upper half-plane, which is equipped with the hyperbolic metric. This metric allows us to measure distances and angles in a way that reflects the unique properties of hyperbolic geometry.

We define the hyperbolic area $\mu(A)$ of a subset A in the upper half-plane using the following double integral:

$$\mu(A) = \iint_A \frac{dx dy}{y^2}$$

This integral exists under certain conditions. Our next objective is to demonstrate that an isometry of the upper half-plane preserves area. Specifically, if we consider an element of the projective special linear group, $\text{PSL}(2, R)$, it will also preserve the area.

To accomplish this, we will need to invoke a fundamental fact: an analytic function satisfies the Cauchy-Riemann equations.

Let's recall what the Cauchy-Riemann equations are. For an analytic function $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, the function must satisfy the following conditions:

$$1. \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$2. \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These equations are essential in complex analysis, and I'm sure you are already familiar with them, so I will not delve into a proof.

Now, let's state the theorem: Suppose we have a subset A of the upper half-plane such that $\mu(A)$ exists. Let T be an element of $\text{PSL}(2, R)$. Then, we have:

$$\mu(T(A)) = \mu(A)$$

To prove this, we can express T in the form:

$$T(z) = \frac{az + b}{cz + d}$$

where $ad - bc = 1$ and a, b, c, d are real numbers. If we let $z = x + iy$, we can rewrite $T(z)$ as $T(z) = u(x, y) + iv(x, y)$. Importantly, since T is a transformation of this form, it is also an analytic function.

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T is an analytic function. T satisfies Cauchy-Riemann Equations i.e. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

The Jacobian $\frac{\partial(u, v)}{\partial(x, y)} \stackrel{\text{defn}}{=} \det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

$$= \left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|^2 \quad \text{--- (1)}$$

$T(z) = u + iv$
 $T'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$\frac{\partial(u, v)}{\partial(x, y)} = |T'(z)|^2$$

$$= \frac{1}{|cz + d|^4}$$

$T(z) = \frac{az + b}{cz + d}$
 $T'(z) = \frac{a(cz + d) - (az + b)c}{(cz + d)^2}$
 $= \frac{1}{(cz + d)^2}$

Therefore, the transformation T satisfies the Cauchy-Riemann equations. Specifically, we have:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Now, if we compute the Jacobian of this transformation, we denote it as follows:

$$J = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

By definition, the Jacobian is the determinant of this matrix, which can be expressed as:

$$J = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x}.$$

Substituting the Cauchy-Riemann equations into this expression, we find that:

$$J = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2.$$

This can also be expressed as the square of the modulus of the complex number $\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$, which yields:

$$J = |T'(z)|^2,$$

where $T'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$. I'm sure you are familiar with this computation from your complex analysis course.

Now, referring back to the earlier equation, we conclude that the Jacobian is equal to:

$$J = |T'(z)|^2.$$

Next, we also need to express the differential dz in terms of the transformation T :

$$dz = \frac{Az + B}{Cz + D}.$$

To compute $dT(z)$, we differentiate this expression. The derivative of the denominator $Cz + D$ is simply C , so we have:

$$dT(z) = \frac{Cz + D}{(Cz + D)^2}.$$

Now, taking the derivative of the numerator, we find that it can be expressed as:

$$A(Cz + D) - (Az + B)C.$$

Given that $ad - bc = 1$, the numerator simplifies to:

1.

Thus, we conclude that:

$$J = \frac{1}{|Cz + D|^4}$$

Lastly, I would like to remind you that we previously expressed dz as $u + i b$.

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$T(z) = u + iv$

$\mu(T(A)) = \int_{T(A)} \frac{du dv}{v^2} = \int_A \frac{\partial(u, v)}{\partial(x, y)} \frac{dx dy}{v^2}$

$T: \mathbb{H}^2 \rightarrow \mathbb{H}^2$

$= \int_A \frac{1}{|cz + d|^4} \frac{dx dy}{v^2} \quad (*) \quad \frac{du dv}{v^2} = \frac{\partial(u, v)}{\partial(x, y)} dx dy$

$v = \text{Im}(T(z))$

$T(z) = \frac{az + b}{cz + d} = \frac{(az + b)(\overline{cz + d})}{|cz + d|^2}$

$= \frac{(az + b)(c\bar{z} + d)}{|cz + d|^2}$

$z = x + iy$

$T(z) = \frac{a(x + iy)^2 + b + bcx + ndx + iy(ad - bc)}{|cz + d|^2}$

$v = \text{Im}(T(z)) = \frac{y}{|cz + d|^2} \quad ad - bc = 1$

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Therefore, we can express the hyperbolic area $\mu(T(A))$ as the double integral over the transformed region $T(A)$:

$$\mu(T(A)) = \iint_{T(A)} \frac{du dv}{v^2}$$

This means that T is a mapping from the upper half-plane to itself. According to the change of variables formula, this integral can be rewritten as:

$$\mu(T(A)) = \iint_A |J| \frac{dx dy}{v^2},$$

where $|J|$ is the Jacobian of the transformation. We previously computed the Jacobian to be:

$$|J| = \frac{1}{|Cz + D|^4}.$$

Now, let's consider v , which represents the imaginary part of dz . The expression for dz is given by:

$$dz = \frac{Az + B}{Cz + D}.$$

We can further analyze this by expressing dz in terms of its modulus:

$$dz = \frac{Az + B}{|Cz + D|^2}.$$

To compute v^2 , we must consider the imaginary part of dz . This leads to:

$$v = \text{Im}(dz) = \frac{y}{|Cz + D|^2}.$$

Here, $ad - bc = 1$ since we are dealing with an element of $PSL(2, R)$, which ensures the determinant condition holds. Therefore, from this equation, we can write:

$$\mu(T(A)) = \iint_A \frac{1}{|Cz + D|^4} \frac{dx dy}{\left(\frac{y}{|Cz + D|^2}\right)^2}.$$

This simplifies to:

$$\mu(T(A)) = \iint_A \frac{1}{|Cz + D|^4} \cdot \frac{|Cz + D|^4}{y^2} dx dy.$$

As a result, we find that:

$$\mu(T(A)) = \mu(A).$$

This confirms that the hyperbolic area is preserved under the isometry T. Thus, we have demonstrated that the hyperbolic area is invariant under such transformations.

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$$\mu(T(A)) = \int_A \frac{1}{|cz+dt|^2} \frac{|cz+dt|^2}{y^2} dx dy = \int_A \frac{dx dy}{y^2} = \mu(A)$$

Theorem:- (Gauss-Bonnet Area of a geodesic triangle in \mathbb{H}^2)
 Let Δ be a geodesic triangle in \mathbb{H}^2 with vertex angles α, β, γ . Then area of Δ , $\mu(\Delta) = \pi - (\alpha + \beta + \gamma)$

Proof:- Case (1) Let one of the vertices of Δ lie in $\mathbb{R} \cup \{\infty\}$

$T(z) = \frac{-1}{z-2}$
 $T \in \text{Isom}(\mathbb{H}^2)$
 $T \in \text{PSL}(2, \mathbb{R})$

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We have established that the hyperbolic area $\mu(T(A))$ can be expressed as the integral:

$$\mu(T(A)) = \iint_A \frac{1}{|Cz + D|^4} \cdot \frac{dx dy}{\left(\frac{y}{|Cz + D|^2}\right)^2}$$

This simplifies to:

$$\mu(T(A)) = \iint_A \frac{1}{|Cz + D|^4} \cdot \frac{|Cz + D|^4}{y^2} dx dy,$$

which is equivalent to:

$$\mu(T(A)) = \mu(A).$$

This means that if we take T from $PSL(2, R)$, which is an isometry of the upper half-plane, it follows that $\mu(T(A)) = \mu(A)$.

Now, let's move on to the next theorem, which states that the area of a geodesic triangle is equal to π minus the sum of its vertex angles. This theorem is often referred to as the Gauss-Bonnet theorem.

To elaborate, consider a geodesic triangle Δ in the upper half-plane with vertex angles α , β , and γ . The area of this triangle, denoted by $\mu(\Delta)$, can be expressed as:

$$\mu(\Delta) = \pi - (\alpha + \beta + \gamma).$$

In our context, a geodesic triangle refers to a triangle in the upper half-plane where all three sides are geodesics. Let's visualize this concept. Here is the triangle Δ in the upper half-plane, with the vertex angles labeled as α , β , and γ . The theorem asserts that the area of this triangle is given by:

$$\mu(\Delta) = \pi - (\alpha + \beta + \gamma).$$

Now, let's delve into the proof of this theorem. In Case 1, we will assume that one of the vertices of triangle Δ lies on the real axis or possibly at infinity. For our purposes, let's say one of the vertices lies on the real line. The configuration would look like this, where one vertex is positioned on the real line at a point x .

Initially, we will demonstrate that the area of this triangle is $\pi - (\alpha + \beta)$. The real line intersects with our triangle at vertex x . To facilitate our proof, we will apply the transformation defined as:

$$T(z) = -\frac{1}{z - x}.$$

It is important to note that this transformation T is indeed an isometry of the upper half-plane and belongs to $PSL(2, R)$. Consequently, T will preserve the area of triangle Δ .

Upon applying this transformation, we can observe the resulting triangle where $T(x)$ maps

to infinity. Since T is a conformal map, it will also preserve angles. We can label the sides of our original triangle as γ_1 , γ_2 , and γ_3 . The image of the original triangle under the transformation T will produce a vertical line, given that T maps x to infinity.

As we visualize this, let's denote the angles at the vertices of the transformed triangle. The angle at the original vertex α remains unchanged under the transformation, while the angle β is similarly preserved. Furthermore, we can denote the center of the semicircle defined by the transformation as point A .

At this point, we have established a solid foundation for understanding the relationship between the area of a geodesic triangle and its vertex angles, demonstrating how transformations can help in the analysis of hyperbolic geometry.

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Let $S(z) = z - a$

$S(z) = z - a$

$area(\Delta) = area(ST(\Delta))$

Let the equation of the semi-circle containing the geodesic segment AB to be $y = \sqrt{r^2 - x^2}$ $r = |OA| = |OB|$ (Euclidean length)

The equation of the geodesic segment AB is $y = \sqrt{r^2 - x^2}$, $x \in [a, b]$ for some $a, b \in \mathbb{R}$

$$u(\Delta) = u(ST(\Delta)) = \int_{ST(\Delta)} \frac{dx dy}{y^2}$$

$$= \int_a^b \int_{\sqrt{r^2 - x^2}}^{\infty} \frac{dx dy}{y^2}$$

Let's define $S(z) = z - a$. When we apply this transformation, the figure will be altered accordingly. Specifically, if I use $S(z) = z - a$, the previous configuration will transform into a similar image, but now the center of the semicircle will be located at the origin.

Now, if we take the composition $S \circ T$ of the lines γ_3 , γ_1 , and γ_2 , we see that this transformation preserves the structure of the triangle $S \circ T(\Delta)$. It is important to note that S is also an isometry and belongs to $\text{PSL}(2, R)$. As a result, the area of triangle Δ remains unchanged under this transformation, leading us to conclude that the area of Δ is equal to the area of $S \circ T(\Delta)$.

Next, let us focus on determining the area of the triangle whose vertex angles are α and β . We aim to show that the area can be expressed as $\pi - (\alpha + \beta)$.

For clarity, let's denote the vertices of our triangle as A and B . We will draw vertical lines, designating one line at $x = A$ and another at $x = B$. Now, let's connect points O to B and O to A using straight lines, creating a Euclidean framework.

When we draw a tangent line at point A , the angle between this tangent and the vertical line is α . Similarly, if we draw a tangent at point B , the angle between this tangent and the vertical line will be β . Notably, both of these angles are right angles due to the property that tangents are perpendicular to the radius at the point of contact. Therefore, we can confirm that the angle opposite to A is α and the angle opposite to B is β .

Now, let's compute the area of the triangle by evaluating the integral. The equation of the semicircle that contains the geodesic segment AB can be expressed as:

$$y = \sqrt{r^2 - x^2},$$

where r is the Euclidean distance defined as $|OA|$ or $|OB|$. Hence, we have the geodesic segment AB represented by the equation $y = \sqrt{r^2 - x^2}$, with x varying from a to b .

Next, the hyperbolic area $\mu(\Delta)$ can be represented by the following integral:

$$\mu(\Delta) = \iint \frac{dx dy}{y^2},$$

where the limits for x range from a to b , and the limits for y extend from $\sqrt{r^2 - x^2}$ to ∞ . Thus, we need to compute the area, and we can express this as:

$$\mu(\Delta) = \iint_A \frac{dx dy}{y^2},$$

which leads us to evaluate the double integral from a to b and from $\sqrt{r^2 - x^2}$ to ∞ . In doing so, we establish that the hyperbolic area $\mu(\Delta)$ can indeed be computed based on the specified limits and yields the desired result.

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$$\mu(\Delta) = \int_a^b dx \int_{\sqrt{r^2 - x^2}}^{\infty} \frac{dy}{y^2}$$

$$= \int_a^b \frac{dx}{\sqrt{r^2 - x^2}}$$

Put $x = r \cos \theta$, $\pi - \alpha \leq \theta \leq \beta$

$$= \int_{\pi - \alpha}^{\beta} \frac{-r \sin \theta d\theta}{r \sin \theta} = \pi - (\alpha + \beta)$$

area of $\Delta = \pi - (\alpha + \beta)$

Case (ii) :- Let Δ be a triangle in \mathbb{H}^2 with vertex angles α, β, γ

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We can express this as:

This integral can be rewritten as follows:

$$\int_a^b dx \int_{\sqrt{r^2 - x^2}}^{\infty} \frac{dy}{y^2}.$$

By simplifying this expression, we find that it is equivalent to:

$$\int_a^b \frac{dx}{\sqrt{r^2 - x^2}}$$

Now, if we perform the substitution $x = r \cos \theta$, where θ varies from $\pi - \alpha$ to β , the integral transforms into:

$$\int_{\pi-\alpha}^{\beta} (-r \sin \theta) \cdot \frac{-r \sin \theta d\theta}{r \sin \theta}.$$

This simplifies to the integral:

$$\int_{\pi-\alpha}^{\beta} d\theta,$$

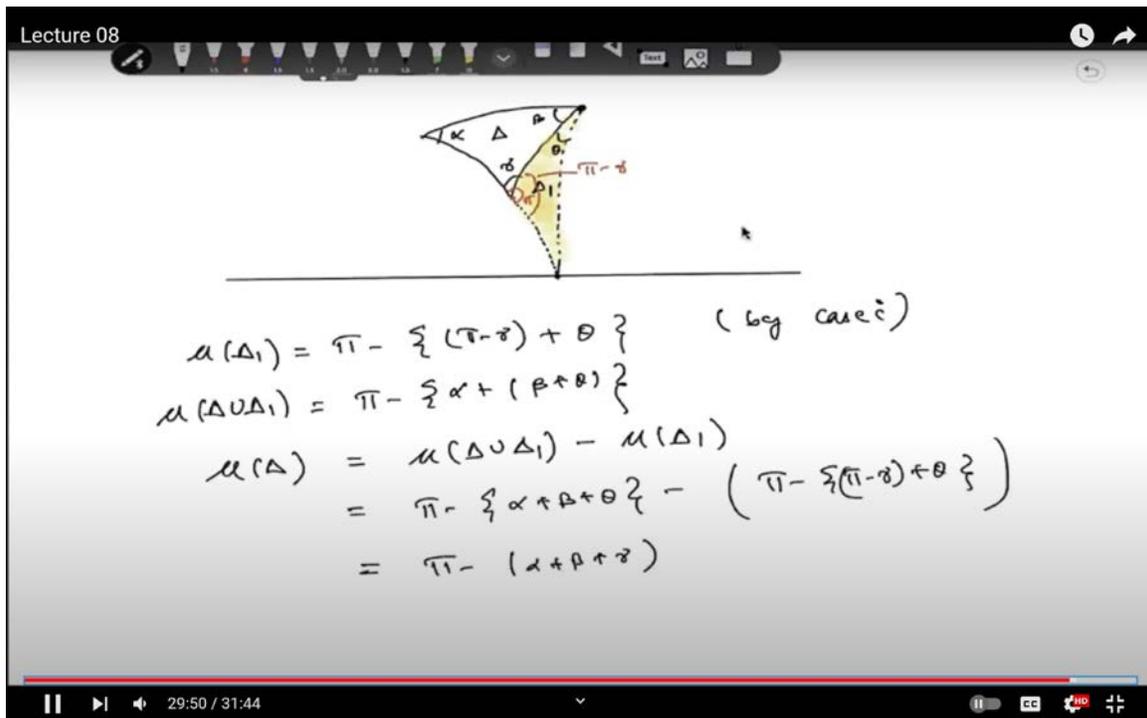
which results in:

$$\pi - \alpha + \beta.$$

Thus, we conclude that the area of this triangle Δ in this specific scenario is given by:

$$\mu(\Delta) = \pi - \alpha + \beta.$$

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Diagram showing a triangle with angles α , β , and γ . A shaded region is shown, and the angle $\pi - \beta$ is indicated.

Handwritten derivations:

$$\begin{aligned} \mu(\Delta_1) &= \pi - \left\{ (\pi - \beta) + \theta \right\} \quad (\text{by case i}) \\ \mu(\Delta \cup \Delta_1) &= \pi - \left\{ \alpha + (\beta + \theta) \right\} \\ \mu(\Delta) &= \mu(\Delta \cup \Delta_1) - \mu(\Delta_1) \\ &= \pi - \left\{ \alpha + \beta + \theta \right\} - \left(\pi - \left\{ (\pi - \beta) + \theta \right\} \right) \\ &= \pi - (\alpha + \beta + \theta) \end{aligned}$$

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Now, let us move on to Case 2, which presents a more general situation. In this case, we consider a triangle Δ in hyperbolic space H^2 with vertex angles α , β , and γ . We will utilize the findings from Case 1 to demonstrate that:

$$\mu(\Delta) = \pi - \alpha + \beta + \gamma.$$

To visualize this, let's draw a diagram representing the triangle in hyperbolic space.

Consider this geodesic triangle, where each of the sides represents a geodesic. In this triangle, we have angles measuring π , β , and 0 for γ . Now, the next step is to extend one of the sides until it intersects the real axis. After doing this, we connect the two intersection points with another geodesic. This construction yields two triangles: one is the original triangle, denoted as Δ , and the other, formed by this new intersection, we will call Δ_1 .

To clarify, let's label the first triangle as Δ and the second triangle, which is shaded yellow, as Δ_1 . Now, it's important to note that Δ_1 is the triangle discussed in Case 1, and the union of Δ and Δ_1 also represents a triangle considered in Case 1.

Now, observe that the total angle formed at the vertex of Δ_1 will be π . Therefore, one of the vertex angles of Δ_1 can be expressed as $\pi - \gamma$. We can denote this angle as θ .

Thus, we can express the area of Δ_1 as follows:

$$\mu(\Delta_1) = \pi - (\text{sum of the vertex angles}),$$

where one vertex angle is $\pi - \gamma$ and the other is θ . By applying the principles from Case 1, we also note that the union of Δ and Δ_1 corresponds to a triangle already covered in Case 1. Therefore, we can write:

$$\mu(\Delta \cup \Delta_1) = \pi - (\text{sum of the vertex angles}) = \pi - (\alpha + \beta + \theta).$$

Consequently, we can express the area of Δ as the difference between the areas of $\Delta \cup \Delta_1$ and Δ_1 :

$$\mu(\Delta) = \mu(\Delta \cup \Delta_1) - \mu(\Delta_1).$$

Substituting our earlier expressions, we have:

$$\mu(\Delta) = (\pi - \alpha - \beta - \theta) - (\pi - (\pi - \gamma + \theta)).$$

This simplifies to:

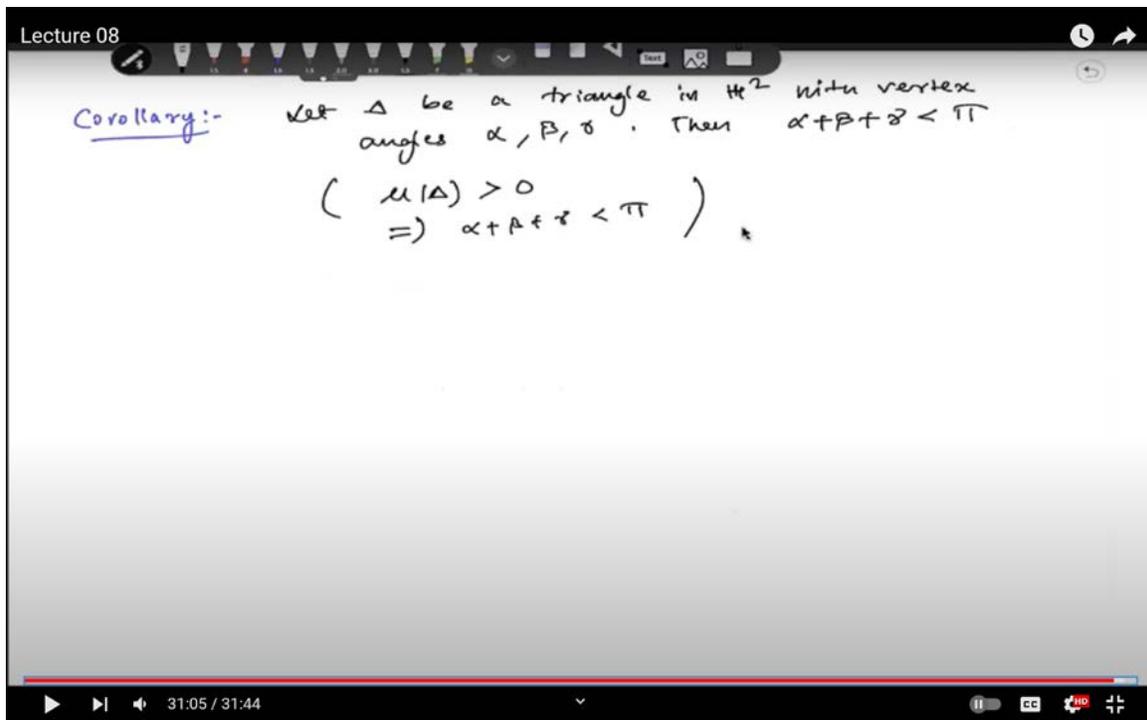
$$\mu(\Delta) = \pi - \alpha - \beta - \gamma.$$

Thus, we have successfully proven that the area of a geodesic triangle is given by:

$$\mu(\Delta) = \pi - (\alpha + \beta + \gamma).$$

And that concludes our proof.

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As a corollary, we can state the following: Let Δ be a triangle situated in the upper half-plane, characterized by vertex angles α , β , and γ . In this case, the sum of the angles $\alpha + \beta + \gamma$ is less than π .

This statement holds true because the area of the triangle, denoted as $\mu(\Delta)$, is always greater

than zero. Consequently, this implies that the sum of the angles must be less than π . To put it succinctly, if $\mu(\Delta) > 0$, then it follows that $\alpha + \beta + \gamma < \pi$.

In contrast, this fact does not apply to Euclidean triangles, where the sum of the angles is exactly equal to π . Okay, I will stop here.