An Introduction to Hyperbolic Geometry

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Module - 3

Lecture - 7

Inner Product and Angles Between Geodesics in Hyperbolic Geometry

Welcome to this lecture on hyperbolic geometry! In today's session, we will explore the concept of the inner product on the upper half-plane concerning the hyperbolic metric. Additionally, we will examine the angle between two geodesics that meet at a single point.

To begin, in order to define the inner product, we first need to establish a vector space. Specifically, we will define the tangent space of the upper half-plane, which will serve as our vector space. This tangent space is isomorphic to R^2 as a vector space.

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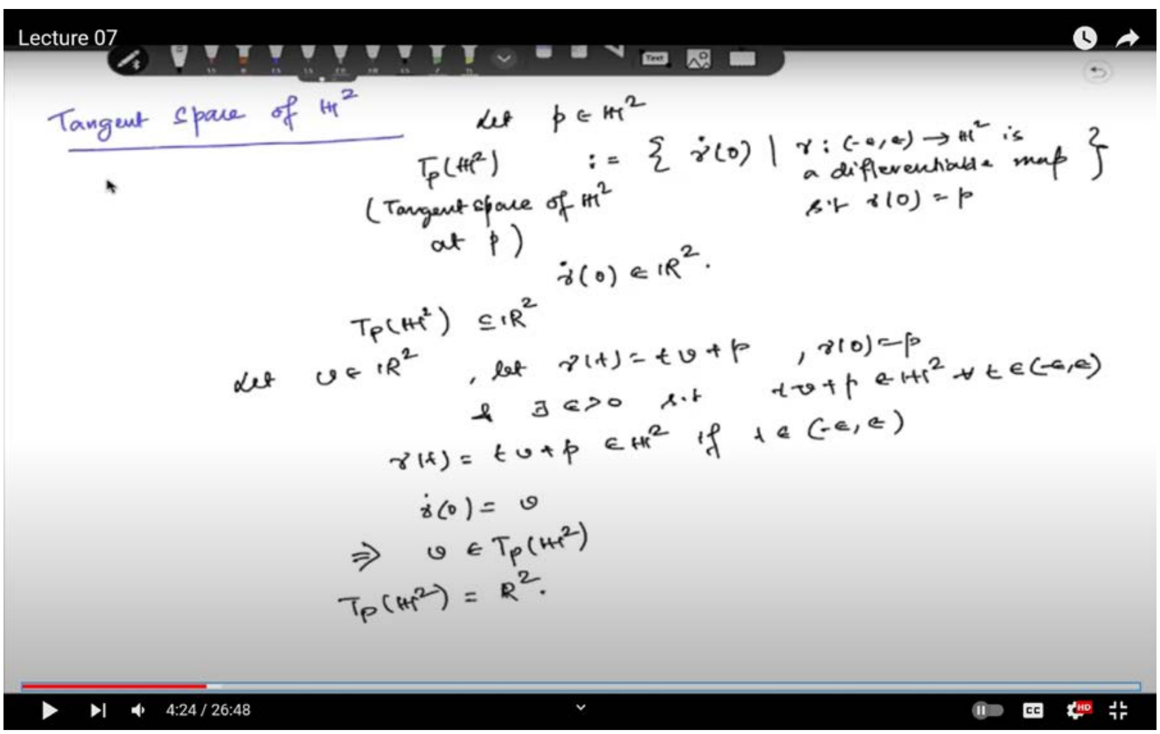

Let me begin by defining the tangent space of the upper half-plane. Consider a point p in the upper half-plane. The tangent space at the point p, denoted as $T_p H^2$, is defined as follows: it is the set of all velocity vectors $\dot{\gamma}(0)$, where γ(t) is a differentiable path in H^2 such that $\gamma(0) = p$. In simpler terms, this tangent space consists of all the velocity vectors corresponding to differentiable paths that pass through the point p. Consequently, $T_p H^2$ forms a vector space.

It is important to note that each $\dot{\gamma}(0)$ lies in R^2. Therefore, we can conclude that $T_p H^2$ is indeed a subset of R^2.

Now, let's take any vector v that belongs to R^2 and consider the path defined by γ(t) = tv + p. Here, γ(0) = p. We can find a small ϵ > 0 such that the image of γ(t), which is tv + p, remains within H^2 for all t in the interval (-ϵ, ϵ).

Since γ(t) is a differentiable path and $\dot{\gamma}(0)$ corresponds to the velocity vector at p, we see that this vector v indeed belongs to $T_p H^2$. Therefore, we can conclude that the tangent space of the upper half-plane is exactly equal to R^2.

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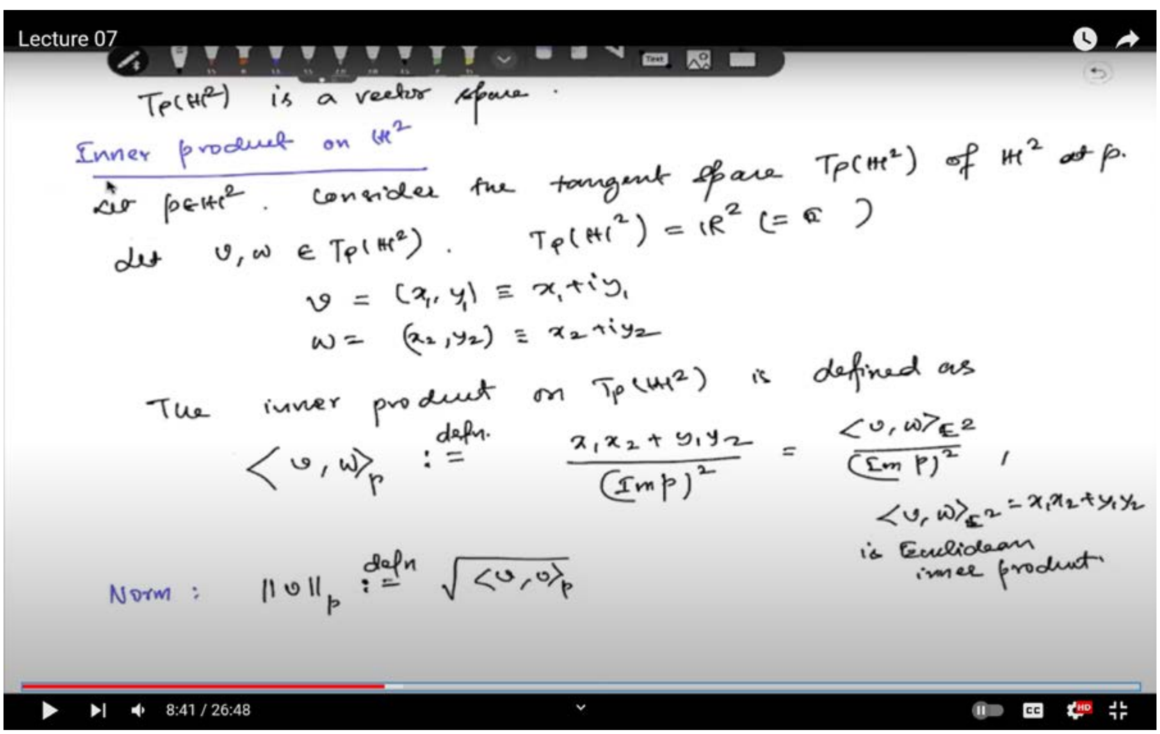

Thus, we can conclude that the tangent space of the upper half-plane is indeed a vector space. Now, let's move on to defining an inner product on this tangent space, which we will refer to as the inner product on this upper half-plane.

This inner product differs from the standard Euclidean inner product, as it is designed to incorporate the hyperbolic metric. Let’s take a point p in H^2 and consider the tangent space at that point, denoted as $T_p H^2$. We will take two vectors within this tangent space. It’s important to note that this tangent space can be represented as R^2 and can also be interpreted as the set of complex numbers.

Let’s denote a vector v as (x_1, y_1) or equivalently as $x_1 + iy_1$. Similarly, we will let another vector w be represented as (x_2, y_2) or $x_2 + iy_2$.

The inner product defined on this tangent space is as follows: it is given by the formula

$$\langle v, w \rangle = \frac{x_1 x_2 + y_1 y_2}{\left(\mathrm{Im}(p)\right)^2},$$

where $\mathrm{Im}(p)$ denotes the imaginary part of the point p. In this definition, we take the standard Euclidean inner product of the vectors v and w and divide it by the square of the imaginary part of the point p.

Now, let’s discuss the norm of a vector. The norm of a vector v is denoted by $|v|$ and is defined as

$$|v| = \sqrt{\langle v, v \rangle}.$$

Next, I will define the angle between two tangent vectors. The inner product we have just defined is distinct from the traditional Euclidean inner product because it has been scaled by the square of the imaginary part of the point p. This scaling captures the unique properties of hyperbolic geometry, allowing us to measure angles and lengths in a way that aligns with the hyperbolic metric.

Let’s discuss the concept of the angle between two vectors, v and w. The angle is defined using the following notation:

$$\theta = \cos^{-1}\left(\frac{\langle v, w\rangle}{|v||w|}\right).$$

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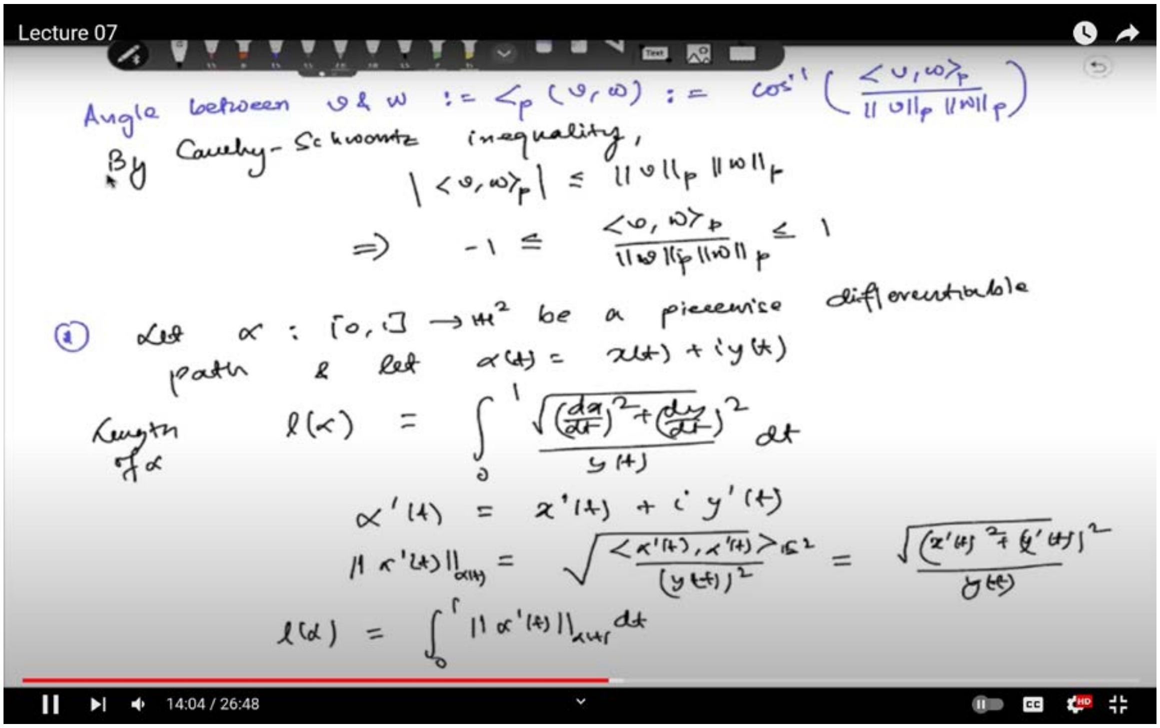

Now, why do we use the inverse cosine function here? This can be justified by the Cauchy-Schwarz inequality. According to this inequality, we have

$$|\langle v, w\rangle| \leq |v||w|.$$

This implies that:

$$-1 \leq \frac{\langle v, w\rangle}{|v||w|} \leq 1.$$

Thus, the expression we are using for the angle is well-defined, allowing us to compute the angle between the two vectors.

Next, we will prove that for an isometry of the upper half-plane, the norm of v is equal to the norm of dT_p, where dT is the differential of the map T.

Before proceeding with the proof, let’s make an important observation. Consider a piecewise differentiable path α defined by

$$\alpha(t) = x(t) + iy(t).$$

The length of this path, denoted as L(α), can be expressed as

$$L(\alpha) = \int_0^1 \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y(t)}\, dt.$$

Now, note that if we take the derivative α'(t), it can be written as

$$\alpha'(t) = x'(t) + iy'(t).$$

The norm of this derivative, considering the hyperbolic inner product, is given by

$$|\alpha'(t)| = \sqrt{\frac{\langle \alpha'(t), \alpha'(t)\rangle}{\left(y(t)\right)^2}}.$$

In this case, the hyperbolic inner product at the point α(t) is computed using the standard Euclidean inner product and divided by the square of the imaginary part, which corresponds to y(t). Thus, we can express the norm as:

$$|\alpha'(t)| = \frac{\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}}{y(t)}.$$

Consequently, the length of the path α can be rewritten as:

$$L(\alpha) = \int_0^1 |\alpha'(t)|\, dt.$$

In summary, the length of the path is determined by the integral of the norm of its derivative, scaled appropriately according to the hyperbolic metric.

Now, let's consider an isometry. Let T be an isometry of the upper half-plane, and define

T(z) to be u + iv, where z = x + iy, with u and v both being functions of x and y.

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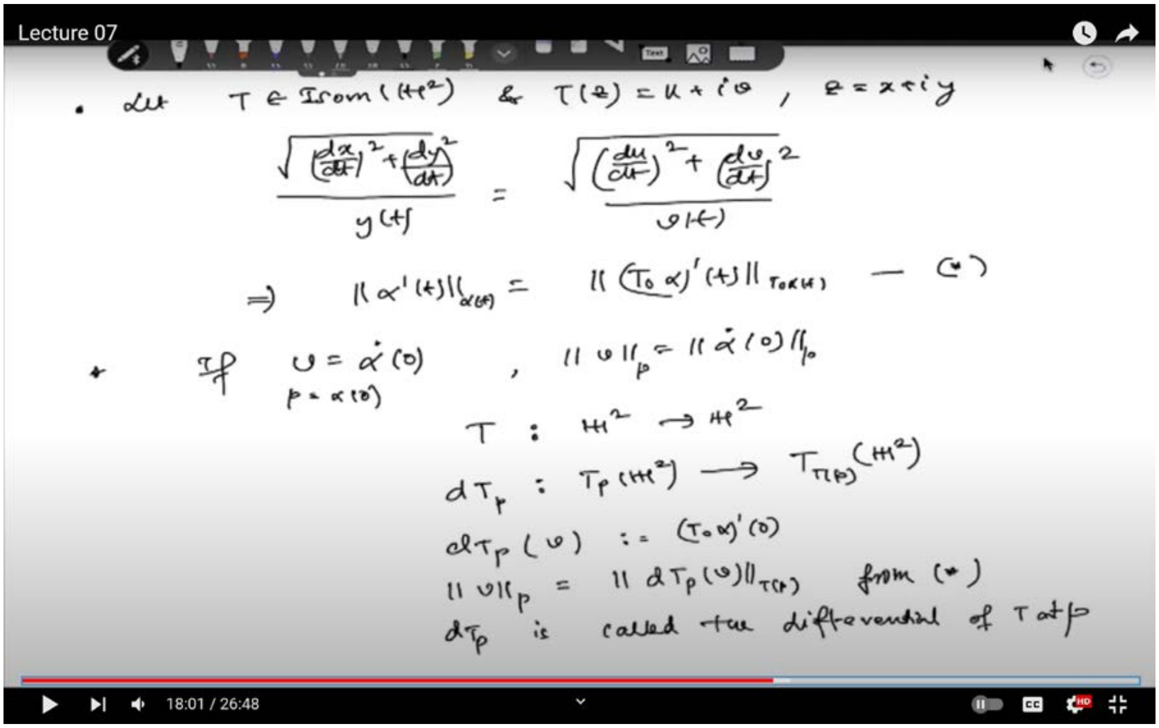

Since T is an isometry, the following relationship will be preserved. We previously established that:

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \div y_t = \sqrt{\left(\frac{du}{dt}\right)^2 + \left(\frac{dv}{dt}\right)^2} \div v_t.$$

This implies that the norm of α'(t) is equal to the norm of dT ∘ α'(t). Now, let's take v to be equal to α'(0) for some differentiable path α. Consequently, the norm of v will be equal to the norm of α'(0).

Let me denote the point p as α(0). In this case, the left-hand side corresponds to the norm of α'(t) evaluated at the point α(t), which equals the norm of dT ∘ α'(t) evaluated at dT ∘ α(t).

Now, the map T operates from the upper half-plane to itself. The differential of T, denoted as dT_p, maps the tangent space of the upper half-plane at the point p to the tangent space at the point T(p) in the upper half-plane. Therefore, we can express the differential $dT_p(v)$ as:

$$dT_p(v) = dT \circ \alpha'(0).$$

As a result, the norm of v at the point p can be represented as:

$$|v|_p = |dT_p(v)|.$$

This follows from the earlier equation. If we substitute t = 0 into this equation, we arrive at:

dT_p is the differential of T at the point p.

Now that we've established this framework, let us proceed to define the next concepts.

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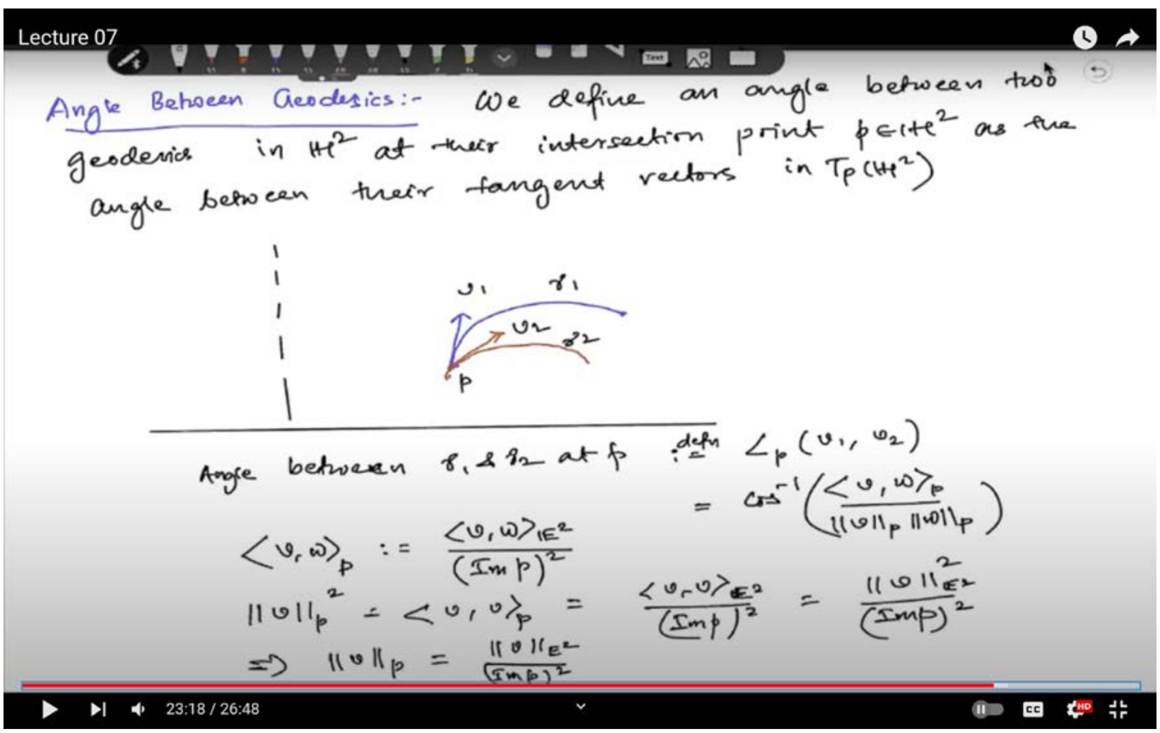

Let’s discuss the concept of the angle between geodesics. When we take two vectors, we have already established a definition for the angle between them. Now, if we consider two geodesics intersecting at a point, we can identify two tangent vectors corresponding to those geodesics at the intersection point. The angle between these tangent vectors will be referred to as the angle between the two geodesics.

Specifically, we define the angle between two geodesics in the upper half-plane, intersecting at the point P (where P belongs to H^2), as the angle between their tangent vectors in the tangent space.

For instance, let's visualize this. If we draw the upper half-plane, it looks something like this. Now, imagine we have two geodesics: one represented by a blue line and the other by a red line. At their intersection, we can assign tangent vectors to each geodesic. Let's denote the tangent vector corresponding to the blue geodesic as v_1 and that of the red geodesic as v_2.

It's important to note that both v_1 and v_2 are vectors in R^2. We can derive the angle between these vectors using the hyperbolic inner product, as well as the Euclidean inner product. If we denote the two geodesics as γ_1 and γ_2, intersecting at the point P, the angle between γ_1 and γ_2 at the point P is defined as the angle between the vectors v_1 and v_2.

Mathematically, this angle can be expressed as:

$$\theta = \cos^{-1}\left(\frac{\langle v_1, v_2\rangle_h}{|v_1| \cdot |v_2|}\right),$$

where $\langle v_1, v_2\rangle_h$ denotes the hyperbolic inner product, and $|v_1|$ and $|v_2|$ are the norms of the vectors.

Now, let's take a closer look at the relationship between the hyperbolic inner product and the Euclidean inner product. The hyperbolic inner product is defined as:

$$\langle v, w\rangle_h = \frac{\langle v, w\rangle_E}{\left(\mathrm{Im}(p)\right)^2},$$

where $\langle v, w\rangle_E$ is the Euclidean inner product. Consequently, the norm of the vector v with respect to the hyperbolic inner product can be expressed as:

$$|v|_h = \frac{|v|_E}{\mathrm{Im}(p)},$$

where $\mathrm{Im}(p)$ refers to the imaginary part of the point p in the upper half-plane.

Thus, we see that the norm of v in the hyperbolic context is determined by scaling its Euclidean norm by the square of the imaginary part of the point p. This relationship allows us to elegantly connect the geometry of the hyperbolic plane with traditional Euclidean concepts, thereby enriching our understanding of angles between geodesics in hyperbolic geometry.

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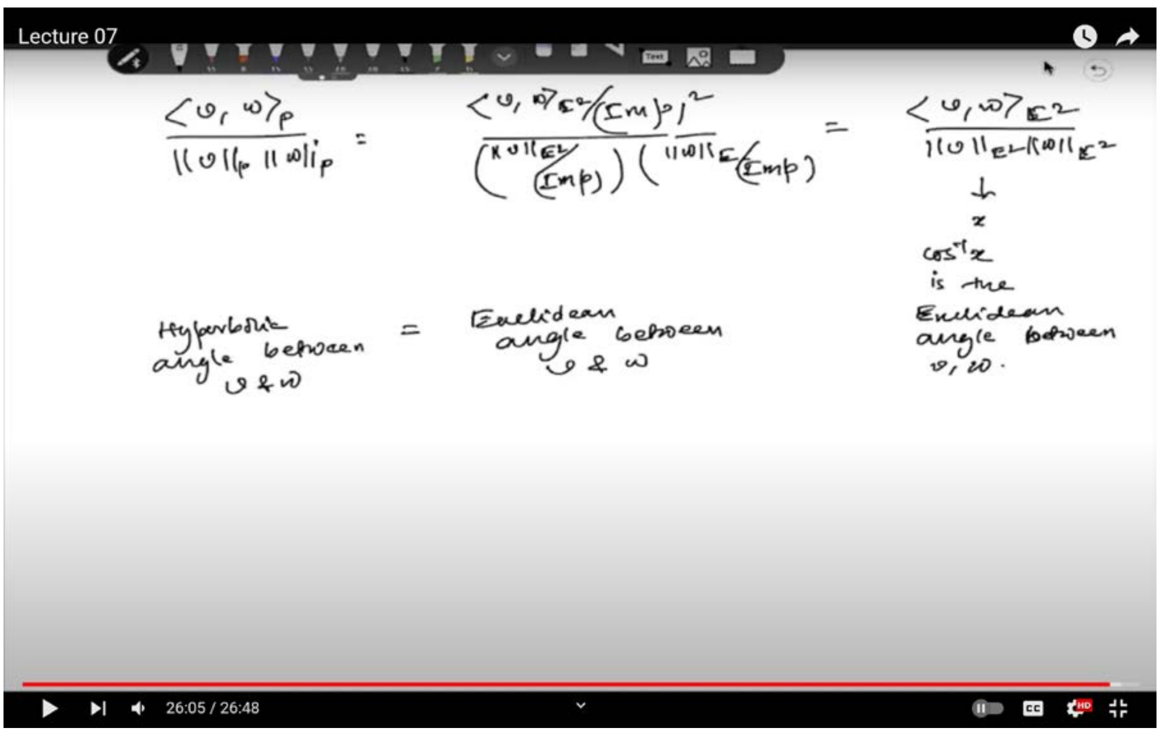

Now, let's explore this important relationship further. We find that the hyperbolic inner product can be expressed by taking the Euclidean inner product and dividing it by the square of the imaginary part of the point P. Specifically, the norm of P can be articulated as follows: it is essentially the Euclidean norm divided by the imaginary part of P, multiplied by the Euclidean inner product of the vectors W and the imaginary part of P. Consequently, this leads us to the realization that the hyperbolic inner product between the vectors V and W can also be formulated as the Euclidean inner product divided by the product of the Euclidean norms of V and W.

In essence, the right-hand side of this equation encapsulates the angle between the vectors V and W from a Euclidean perspective. If we denote this angle as x, we can assert that $\cos^{-1}(x)$ represents the Euclidean angle between the vectors V and W.

Thus, based on this formulation, we have effectively proven that the hyperbolic angle between the vectors V and W is equivalent to the corresponding Euclidean angle.

Moreover, it is noteworthy that we have established that the isometries of the upper half-plane are conformal with respect to the Euclidean angle. This property also holds true for the hyperbolic angle, confirming that the isometries of the upper half-plane are conformal with respect to the hyperbolic angles as well.

With that, I will conclude this segment. In our next class, we will prove that the area of a triangle is equal to π minus the sum of the angles at each of the triangle's vertices.