An Introduction to Hyperbolic Geometry

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Module - 2

Lecture – 6

Isometric Group of the Upper Half-Plane and Its Isomorphism to PSL and Reflections: A Detailed Proof

Hello everyone! Welcome to this lecture on hyperbolic geometry. Today, we will prove an important result: the isometry group of the upper half-plane, with respect to the hyperbolic metric, is isomorphic to the group generated by PSL(2, R) along with reflections across the y-axis. Let's start. To determine the isometry group of the upper half-plane, our first task is to derive an explicit formula for the hyperbolic distance between two points in this space.

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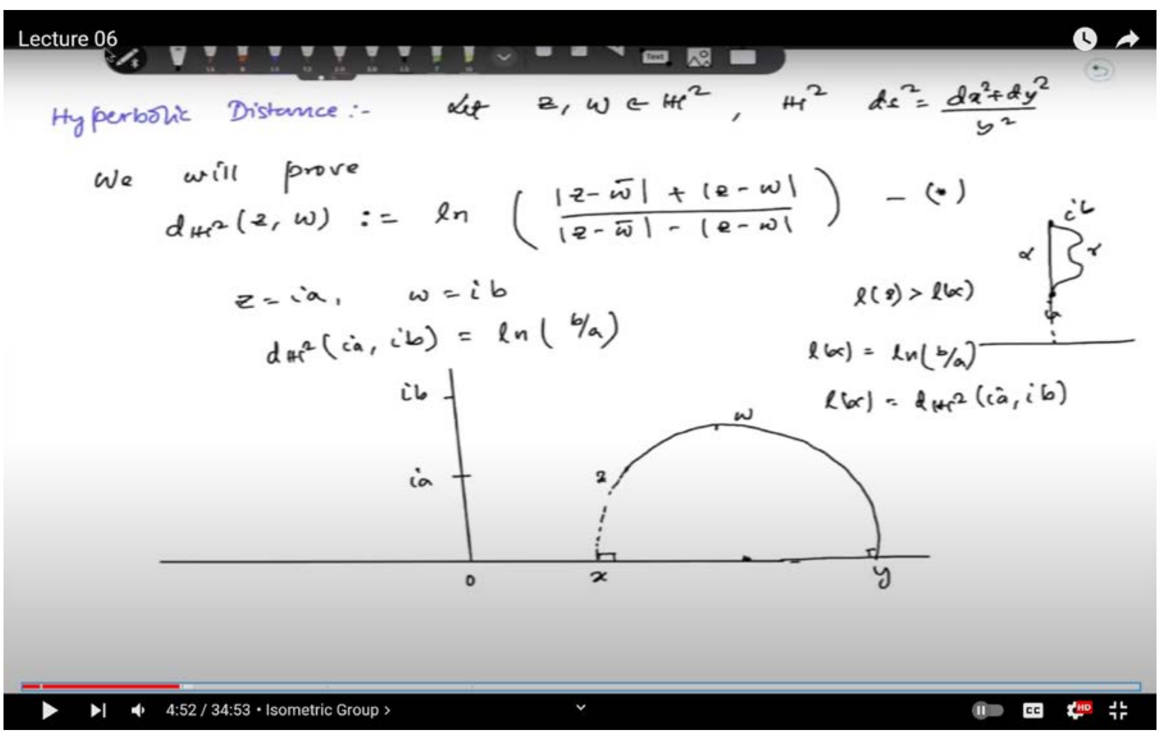

So, let's start by considering two points in the upper half-plane and work our way through the necessary computations to uncover the structure of the isometries in this geometric setting.

Let z and w be two points in the upper half-plane. This upper half-plane is equipped with the hyperbolic metric defined as

$$\frac{dx^2 + dy^2}{y^2}.$$

In this lecture, we will prove the following: the hyperbolic distance between z and w is given by the formula

$$d(z, w) = \log\left(\frac{|z - \overline{w}| + |z - w|}{|z - \overline{w}| - |z - w|}\right).$$

Now, let's take z = i a and w = i b. We have already established that the geodesic joining i a and i b is the imaginary axis. If we denote this vertical geodesic as α, we know that for any other differentiable path γ connecting i a and i b, the length of γ is always greater than or equal to the length of α.

The length of α, which is the geodesic, is precisely

$$\log\left(\frac{b}{a}\right).$$

Thus, for the points z = i a and w = i b, we have already proved that the distance between i a and i b is

$$\log\left(\frac{b}{a}\right).$$

If we substitute z = i a and w = i b into our formula, we find that the right-hand side simplifies to $\log\left(\frac{b}{a}\right)$. Therefore, the equation holds true for z = i a and w = i b.

Next, let's consider any other two points z and w in the upper half-plane that do not lie on the imaginary axis. We have established that if z and w do not lie on a vertical line, there exists a semicircle passing through these points, orthogonal to the real axis, with its center located on the real axis (or x-axis).

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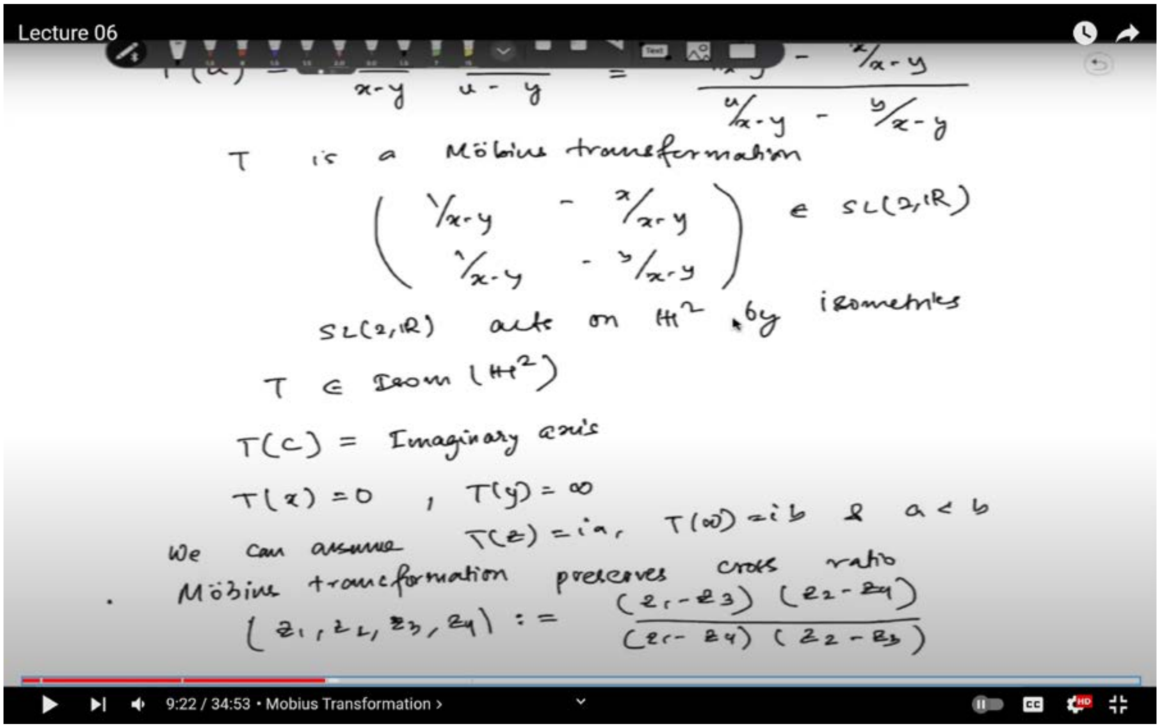

Suppose the semicircle intersects the real axis at points x and y. Now, consider the transformation defined by

$$T(u) = \frac{u-x}{u-y} \cdot \frac{1}{x-y}.$$

This can be rewritten as

$$T(u) = \frac{u}{x-y} - \frac{x}{x-y},$$

and further simplified to

$$T(u) = \frac{u-x}{u-y} \cdot \frac{1}{x-y}.$$

This expression represents a Möbius transformation, and T indeed belongs to $\mathrm{SL}(2, R)$. The matrices in $\mathrm{SL}(2, R)$ act on the upper half-plane through isometries. Thus, we can conclude that T is an isometry of the upper half-plane.

It's also important to note that if we apply T to this semicircle, it transforms it to the imaginary axis. Let's denote this semicircle as C. This property holds because T(x) = 0 and T(y) = ∞. Since T is a Möbius transformation, it maps circles and lines in the complex plane to other circles or lines. Here, as x approaches 0 and y approaches ∞, the transformation T must map this semicircle C to the straight line, which in this case is the imaginary axis.

Therefore, we can assume that T sends i a to i b, where a < b.

Next, we have previously established that Möbius transformations preserve the cross ratio. To clarify, the cross ratio for points z_1, z_2, z_3, z_4 in the complex plane is defined as:

$$[z_1, z_2; z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

This definition illustrates that Möbius transformations maintain the cross ratio. Thus, we can assert that:

$$[T(z_1), T(z_2), T(z_3), T(z_4)] = [z_1, z_2, z_3, z_4].$$

Thus, the isometry T not only preserves distances but also the relationships defined by the cross ratio among points in the complex plane.

Let's consider the points z and w in the context of the complex plane. In this representation, the upper half of the complex plane is denoted as H^{2}, while the lower half can be referred to as $\overline{H^{2}}$. The conjugate of z, represented as $\bar{z}$, resides in the lower half-plane. Now, if we take four points: z, $\bar{z}$, w, and $\bar{w}$, and compute their cross ratio, we must remember that T is a Möbius transformation.

This transformation T extends to the complex plane including infinity, thereby preserving the cross ratio. Specifically, for the points T(z) and T($\bar{z}$), the cross ratio will remain invariant.

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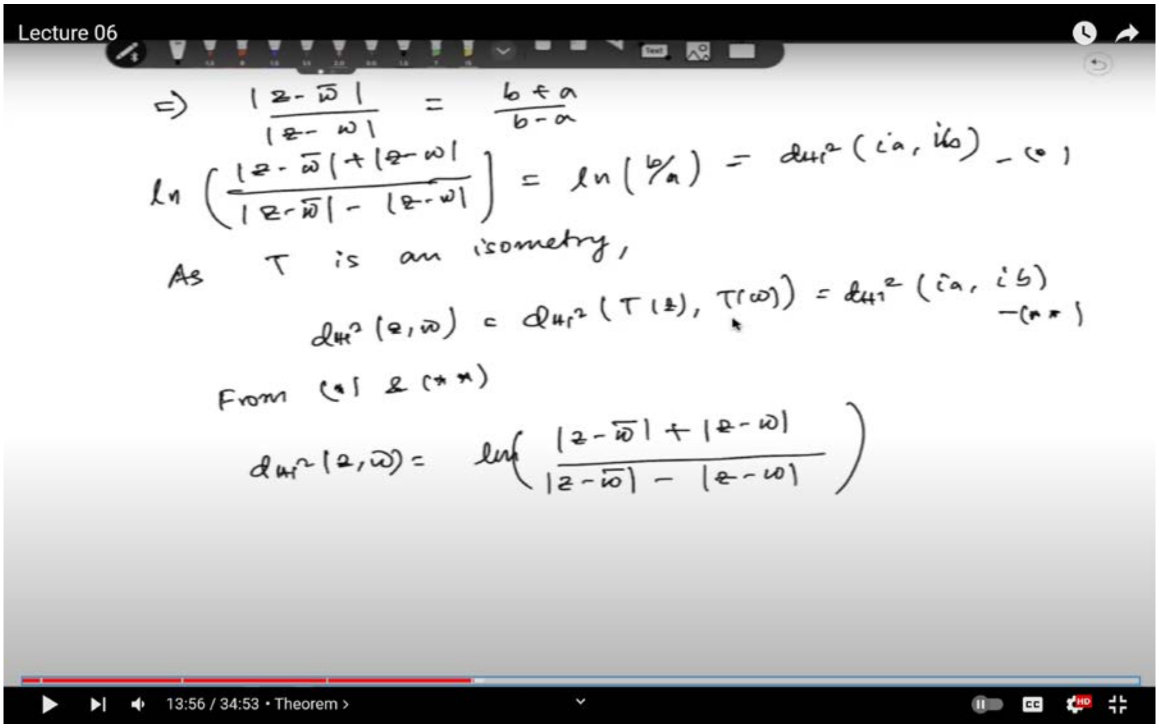

From this relationship, we observe that the points z, w, $\bar{z}$, and $\bar{w}$ are all situated in the complex plane. If we take the modulus of the equation, we can derive the following expression:

$$\frac{|z-w|\cdot|\bar{z}-\bar{w}|}{|w-\bar{z}|\cdot|z-\bar{w}|}=\frac{|a-b|\cdot|a-b|}{|b+a|}.$$

It's important to note that in our previous context, we set T(z) = i a and T(w) = i b. When we take the modulus, we find that |i| = 1, simplifying our calculations. Therefore, we can assert that:

$$\frac{|z-w|}{|w-\bar{z}|}=\frac{b-a}{b+a}.$$

Since b is greater than a, we can safely eliminate the modulus. Consequently, this leads us to:

$$\frac{|z-\overline{w}|}{|z-w|}=\frac{b+a}{b-a}.$$

Now, if we apply the method of componendo and dividendo and take the logarithm of both sides, we obtain:

$$\log\left(\frac{|z-\overline{w}|+|z-w|}{|z-\overline{w}|-|z-w|}\right)=\log\left(\frac{b}{a}\right).$$

This relationship shows that \log \left(\frac{b}{a} \right) is simply the hyperbolic distance between i a and i b, which we have previously established.

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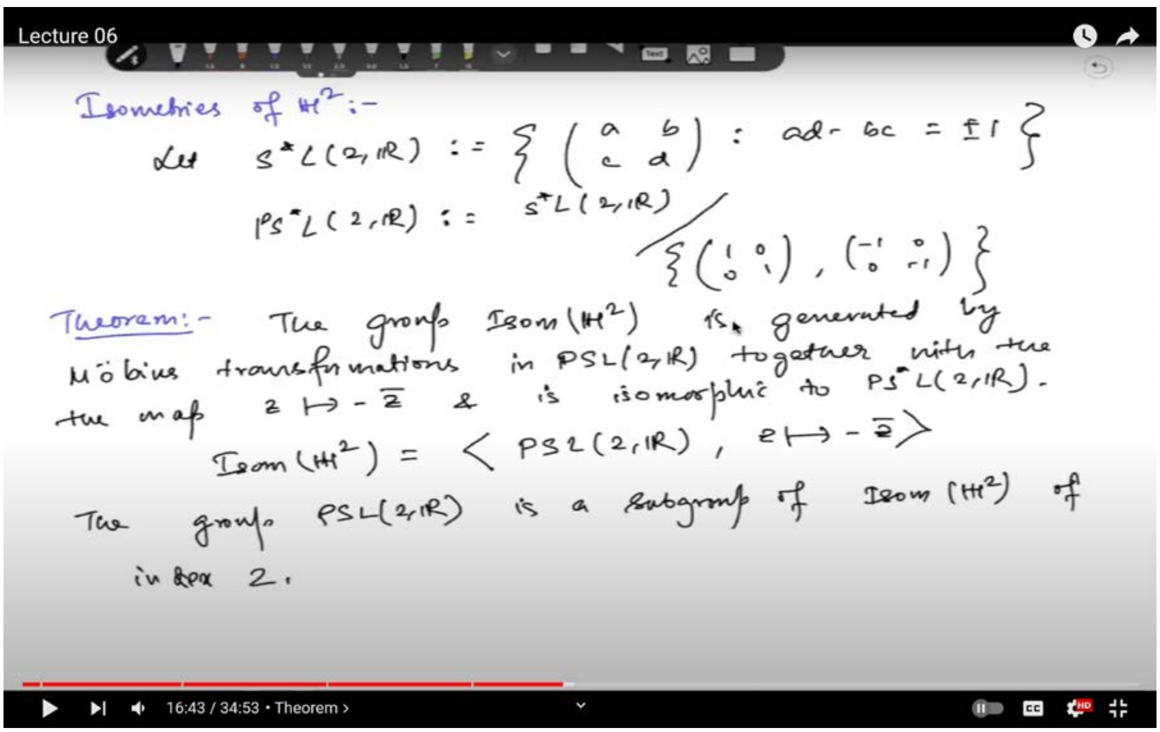

Moreover, since T is an isometry, the distance between z and w is equal to the distance between T(z) and T(w), which corresponds to the distance between i a and i b. Therefore, from these two equations, we conclude that the distance between the points z and w can be expressed as:

$$\text{Distance}(z,w) = \log\left(\frac{|z-\bar{w}| + |z-w|}{|z-\bar{w}| - |z-w|}\right).$$

This elegantly encapsulates the hyperbolic distance in terms of the complex variables.

Now, let's return our focus to the isometries of the upper half-plane. To begin with, I will define the relevant group. Let $\text{SL}(2,R)$ represent the set of all matrices such that the determinant of these matrices is either +1 or -1. In mathematical terms, this means that $\det(A) = AD - BC = \pm 1$.

Next, we consider the quotient group $\text{PSL}(2,R)$, which is essentially $\text{SL}(2,R)$ modulo the equivalence relation that identifies matrices differing only by a factor of ± 1.

Now, let us state the theorem: the isometry group of the upper half-plane is generated by Möbius transformations that belong to $\text{PSL}(2,R)$, in addition to the mapping defined by $z \mapsto -\bar{z}$. In other words, we can conclude that this isometry group is isomorphic to $\text{PSL}(2,R) \times Z/2Z$.

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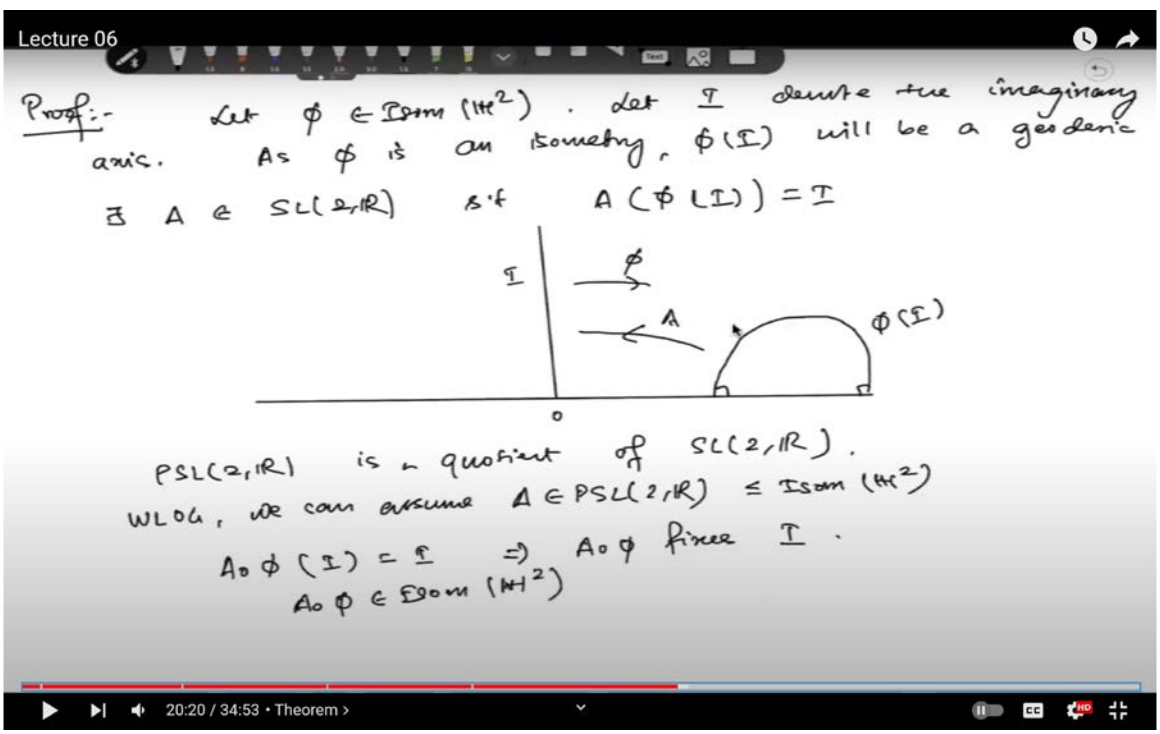

Furthermore, it is important to note that $\mathrm{PSL}(2, R)$ is indeed a subgroup of the isometry group of the upper half-plane, and it has an index of 2 within this larger group. This means that $\mathrm{PSL}(2, R)$ comprises half of the elements in the isometry group of the upper half-plane, clearly illustrating its significant role in the structure of these transformations.

Let us proceed to prove this theorem. We will start by letting φ represent an isometry of the upper half-plane, and let I denote the imaginary axis. Since φ is an isometry, the image of I under φ will also be a geodesic. Specifically, φ(I) will be a bi-infinite geodesic, as I itself is a bi-infinite geodesic.

Now, there exists an element A belonging to $\mathrm{SL}(2, R)$ such that A(φ(I)) = I. This has already been established. So, if we take the imaginary axis I as our reference, we can interpret φ as mapping this imaginary axis to some other geodesic, which we will denote as φ(I).

Importantly, we know that there exists an element A from $\mathrm{SL}(2, R)$ such that $A\big(\phi(I)\big) = I$. This means that A takes the geodesic φ(I) and maps it back to the imaginary axis I. Furthermore, $\mathrm{PSL}(2, R)$ can be viewed as the quotient group of $\mathrm{SL}(2, R)$, allowing us to conclude that, without loss of generality, we can assume A is an element of $\mathrm{PSL}(2, R)$.

Therefore, we have established an element A in $\mathrm{PSL}(2, R)$ that takes φ(I) back to I. Since A maps φ(I) to I, if we consider the composition A ∘ φ, it will fix the imaginary axis I such that A(φ(I)) = I. This implies that A ∘ φ fixes the imaginary axis.

It is also noteworthy that we have previously established that $\mathrm{PSL}(2, R)$ is a subgroup of the isometry group of the upper half-plane. Thus, since A is an isometry and φ is an isometry, the composition A ∘ φ must also be an isometry of the upper half-plane. Consequently, we can conclude that A ∘ φ indeed fixes the imaginary axis I, leading us to our desired results.

Since the composition A ∘ φ fixes the imaginary axis, it will map I to some λ I, where λ is a non-zero scalar. Therefore, we can express this as A ∘ φ(I) = λ I for some $\lambda \neq 0$.

Now, let us define a new transformation $P(z) = \frac{1}{\lambda z}$. This scaling transformation is, in fact, an isometry of the upper half-plane.

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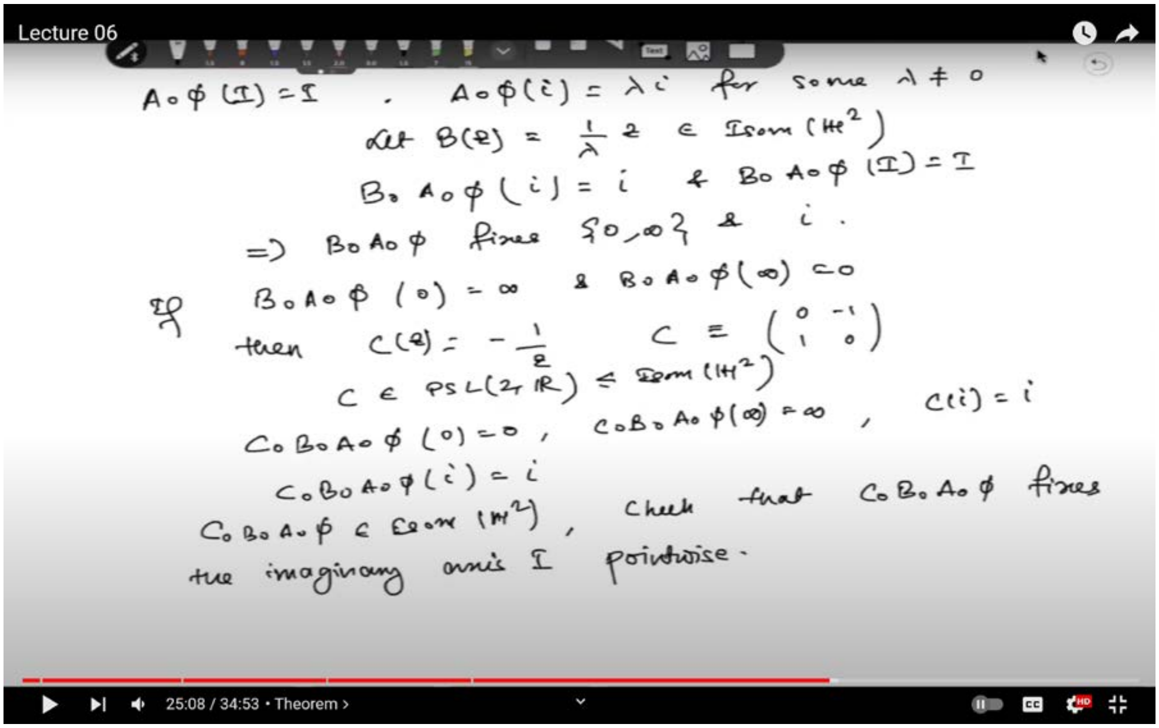

Next, if we consider the composition B ∘ (A ∘ φ), we see that it also fixes the imaginary axis I. This is because B fixes the imaginary axis as well. Thus, we have established that B ∘ (A ∘ φ) fixes the points 0, ∞, and I.

Now, suppose that B ∘ (A ∘ φ) fixes both 0 and ∞. In this case, we can conclude that it can interchange these two points, taking 0 to ∞ and ∞ to 0.

To illustrate this point further, let us define another transformation $C(z) = -\frac{1}{z}$. Corresponding to this transformation C, we have the matrix representation:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The determinant of this matrix is indeed 1, confirming that C is an element of $\mathrm{PSL}(2, R)$ and also an isometry of the upper half-plane.

Now, consider the composition C ∘ (B ∘ A ∘ φ). This isometry will fix 0 and also the point ∞. Since C preserves the imaginary axis, we also find that C(I) = I. Therefore, the composition C ∘ (B ∘ A ∘ φ) will fix the imaginary axis I as well.

Therefore, we started with an isometry φ and pre-composed it with elements from $\mathrm{PSL}(2, R)$, leading us to the conclusion that the resulting composition fixes the imaginary axis I pointwise.

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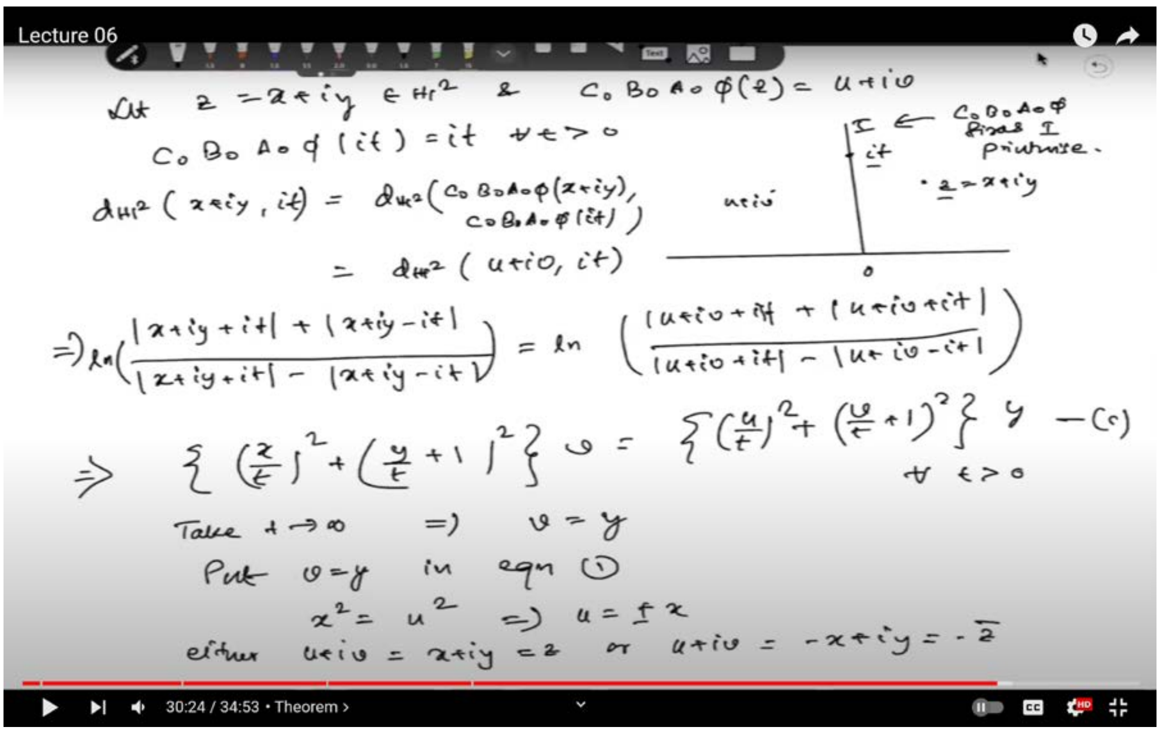

Now, let us consider a point z in the upper half-plane. We will assume that the composition C ∘ B ∘ A ∘ φ(z) can be expressed as u + i b. It is important to note that this isometric transformation, C ∘ B ∘ A ∘ φ, fixes the imaginary axis pointwise. Thus, we have C ∘ B ∘ A ∘ φ(i t) = i t for all t > 0.

As illustrated in the diagram, this is my imaginary axis, denoted as i. The transformation C ∘ B ∘ A ∘ φ indeed fixes the point i pointwise. Now, if we take some point z located above the imaginary axis, we want to demonstrate that the image of this z under our transformation will either equal z itself or be its reflection across the imaginary axis.

Let us express z as x + i y, where x is the real part and y is the imaginary part. We will denote the image of z as u + i v. Our goal is to show that $u = \pm x$ and v = y.

Next, let us calculate the distance between the points z = x + i y and i t. We will apply the isometry C ∘ B ∘ A ∘ φ to these two points, noting that the distance will be preserved. Thus, we can express this distance in terms of u + i v and i t, which is fixed by the isometry.

Now, using the distance formula, we have:

$$\text{Distance} = \frac{|(x+iy)-(it)|+|(x+iy)+(it)|}{|(x+iy)+(it)|-|(x+iy)-(it)|}$$

Taking the logarithm of this expression, we can write:

$$\log\left(\frac{|u+ib+it|+|u+ib-it|}{|u+ib+it|-|u+ib-it|}\right)$$

We find that this logarithmic expression equals the distance between x + i y and i t. Consequently, we can express this as:

$$\log\left(\frac{|u+ib+it|+|u+ib-it|}{|u+ib+it|-|u+ib-it|}\right)$$

From this point, we can remove the logarithm from both sides and perform some calculations, ultimately arriving at the following equation:

$$\frac{x^2}{t^2}+\frac{(y+1)^2}{t^2}v=\frac{u^2}{t^2}+\frac{v^2}{t^2}(y+1)^2$$

This equation holds true for all t > 0.

Now, if we take the limit as t approaches infinity, we can deduce that this implies b = y. Furthermore, substituting b = y back into the equation, we find:

$$x^2 = u^2$$

This leads us to conclude that $u = \pm x$.

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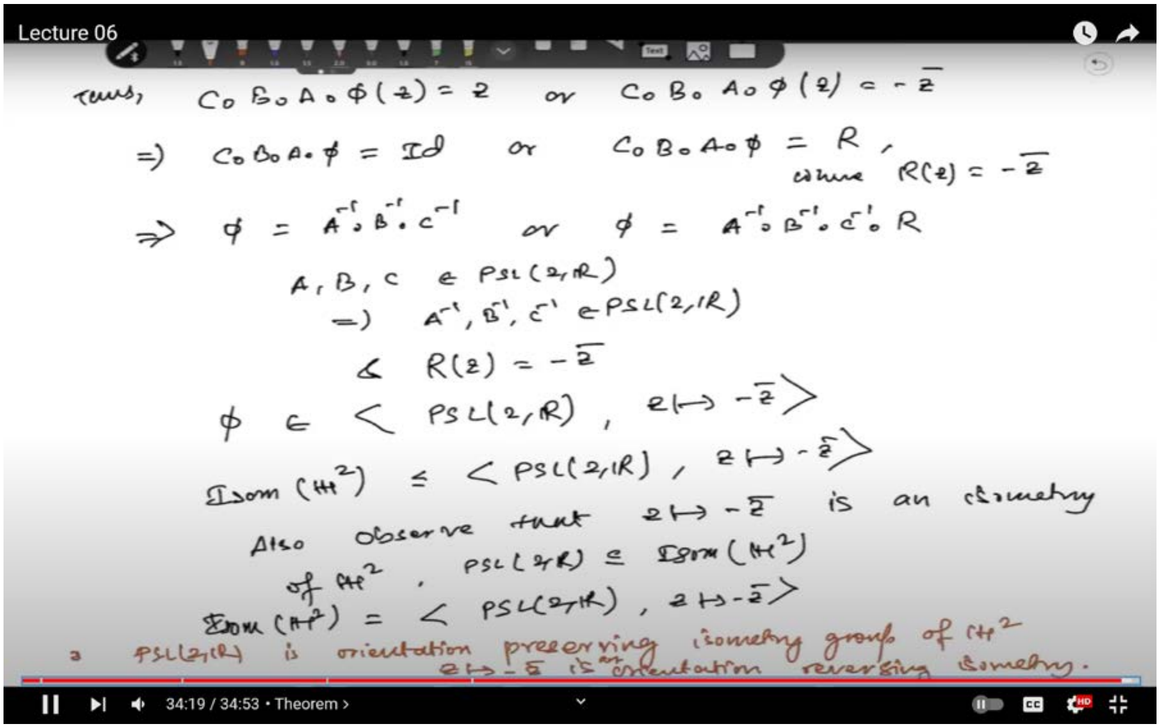

Thus, we have established that either u + ib = x + iy, which is equivalent to z, or u + ib = -x + iy, which corresponds to $-\bar{z}$. Therefore, we can conclude that C ∘ B ∘ A ∘ φ(z) equals either z or $-\bar{z}$. This implies that this composition of isometries can represent either the identity map or a reflection with respect to the imaginary axis.

Consequently, we can express φ in two ways: $\phi = A^{-1} \circ B^{-1} \circ C^{-1}$ or $\phi = A^{-1} \circ B^{-1} \circ C^{-1} \circ R$. Here, A, B, and C belong to the group PSL(2, R), which implies that their inverses also belong to PSL(2, R). The R denotes the reflection with respect to the imaginary axis.

Therefore, we have shown that φ is part of the group generated by PSL(2, R) along with the map $z \mapsto -\bar{z}$. This leads us to conclude that the isometry group of the upper half-plane is indeed a subgroup of PSL(2, R) generated by this reflection.

Moreover, it is noteworthy that the reflection map is itself an isometry of the upper half-plane. We have also established that PSL(2, R) forms a subgroup of the isometries of the upper half-plane. Therefore, we can confidently state that the isometric group of the upper half-plane is generated by PSL(2, R) and the reflection map.

To distinguish between orientation preserving and orientation reversing, we note that PSL$(2, R)$ represents the orientation-preserving isometry group of the upper half-plane, while the map $z \mapsto -\overline{z}$ serves as the orientation-reversing isometry. Okay, so I'll stop here today. Thank you.