An Introduction to Hyperbolic Geometry

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Module - 2

Lecture - 5

Geodesics of the Upper Half Plane with Hyperbolic Metric: An Exploration of Conformal Maps and Möbius Transformations

Welcome to this lecture on hyperbolic geometry! Today, we will explore the geodesics of the upper half-plane in relation to the hyperbolic metric. To determine the geodesics in this upper half-plane, we need to understand an important property: Möbius transformations are conformal, which means they preserve angles. So, let us begin our journey into this fascinating topic.

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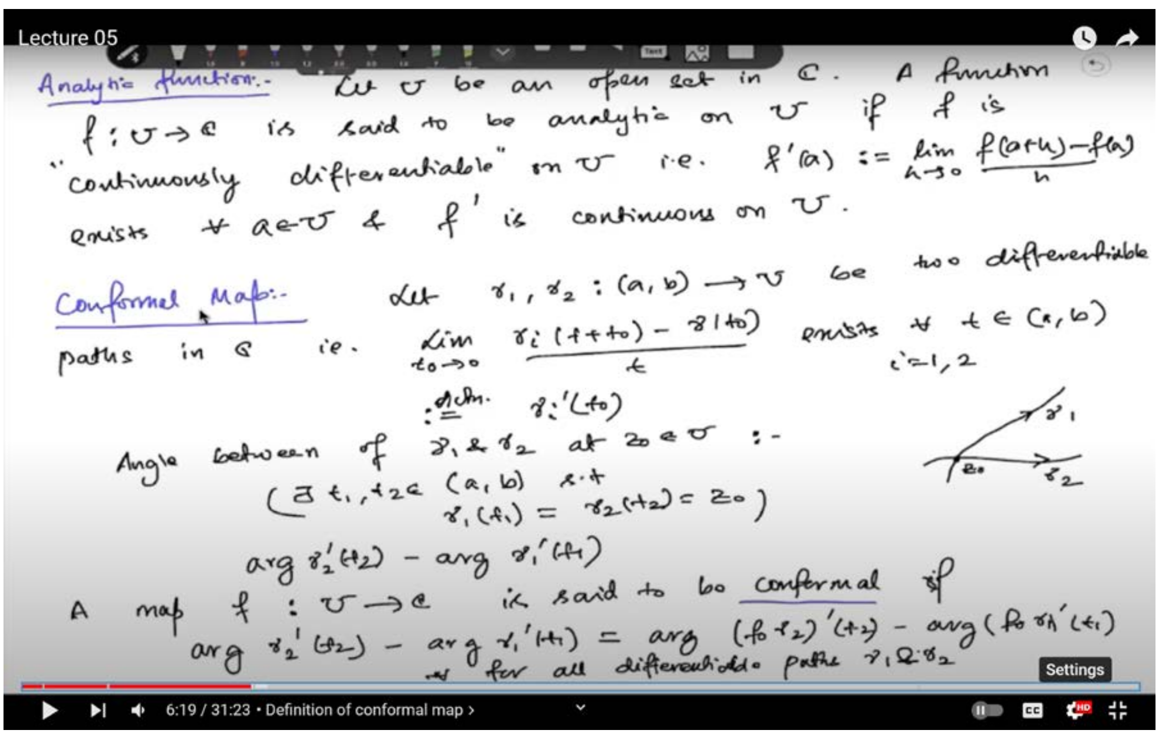

To prove that Möbius transformations are conformal, we will first establish a more general principle: that analytic functions are conformal. But before we dive into that, let's clarify what we mean by an analytic function.

Let us begin by considering an open set, U, within C, the set of complex numbers. Now, a function f, mapping from U to C, is called analytic on U if f is continuously differentiable on U.

What does it mean for a function to be continuously differentiable on U? It means that the derivative of f at any point a in U exists, and this derivative, denoted as f'(a), is given by the limit

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

for all $a \in U$. Additionally, the derivative f' must be continuous across U. This definition of analytic functions should be familiar from complex analysis.

Now, let us move on to define what is meant by a conformal map. To do this, we need to first discuss differentiable paths. Let γ_1 and γ_2 be two differentiable paths in C. A differentiable path means that the following limit exists:

$$\lim_{t_0 \to 0} \frac{\gamma_i(t_0 + t) - \gamma_i(t_0)}{t}, \quad \text{for all } t \in (a, b)$$

where i = 1, 2, and this limit, when it exists, is denoted as $\gamma_i'(t_0)$, the derivative of γ_i at t_0.

Next, we define the angle between two paths γ_1 and γ_2 at a point $z_0 \in U$, where both paths intersect. Suppose that there are points t_1 and t_2 within the open interval such that

$$\gamma_1(t_1) = \gamma_2(t_2) = z_0.$$

The angle between γ_1 and γ_2 at z_0 is defined as the difference between the arguments of their derivatives at that point. Mathematically, it is expressed as:

$$\text{Angle between } \gamma_1 \text{ and } \gamma_2 \text{ at } z_0 = \arg(\gamma_2'(t_2)) - \arg(\gamma_1'(t_1)).$$

Now, a map f from U to C is said to be conformal if it preserves this angle. In other words, for any two differentiable paths γ_1 and γ_2 in U, the angle between them is preserved under the map f. This condition can be written as:

$$\arg\left(f \circ \gamma_2'(t_2)\right) - \arg\left(f \circ \gamma_1'(t_1)\right) = \arg\left(\gamma_2'(t_2)\right) - \arg\left(\gamma_1'(t_1)\right).$$

Thus, f is conformal if this equality holds for all differentiable paths γ_1 and γ_2 on U.

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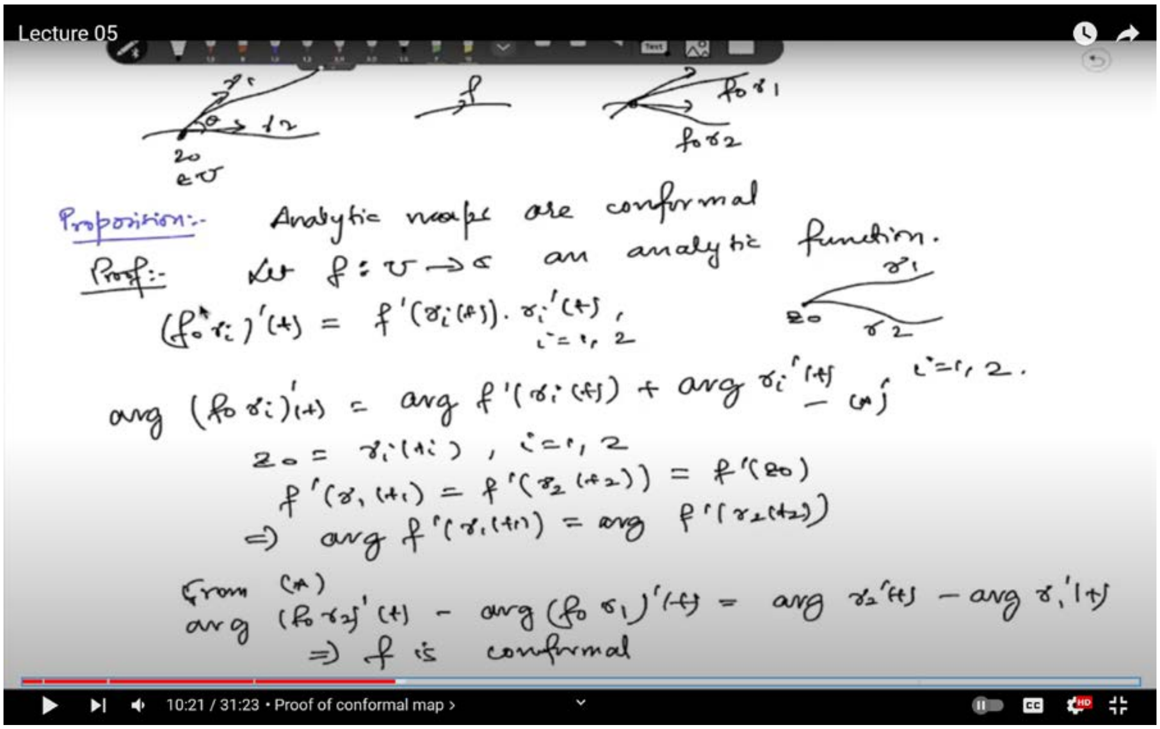

Let's illustrate this concept with a diagram. Suppose we have two paths, γ_1 and γ_2, that intersect at a point $z_0 \in U$. Now, if we apply a function f to these paths, we get new paths, $f \circ \gamma_1$ and $f \circ \gamma_2$. The derivative of γ_1, denoted as γ_1', gives us the tangent vector at z_0, and similarly for γ_2'.

The angle between the two tangent vectors at the intersection point is preserved under the transformation by f. This is the key property of a conformal map. If this condition holds, we call f a conformal map.

Now, let’s make the following important observation: if f is an analytic function, then f is also conformal. Let me state this as a proposition: All analytic maps are conformal.

Proof:

Let f be an analytic function, and consider two paths γ_1 and γ_2 intersecting at z_0. We want to prove that the angle between these two paths is preserved under f, which will confirm that f is conformal.

Since f is analytic, the derivative of the composition of f with the paths γ_1 and γ_2 is given by the product of two terms:

$$f \circ \gamma_i'(t) = f'\big(\gamma_i(t)\big) \cdot \gamma_i'(t)$$

for i = 1, 2, where $f'(\gamma_i(t))$ is the derivative of f, and $\gamma_i'(t)$ is the derivative of the path at t.

Thus, the argument (or angle) of the composition, $\arg\big(f \circ \gamma_i'(t)\big)$, is:

$$\arg\big(f \circ \gamma_i'(t)\big) = \arg\Big(f'\big(\gamma_i(t)\big)\Big) + \arg\big(\gamma_i'(t)\big).$$

Given that $z_0 = \gamma_1(t_1) = \gamma_2(t_2)$, we have $f'\big(\gamma_1(t_1)\big) = f'\big(\gamma_2(t_2)\big) = f'(z_0)$. This implies that the arguments of $f'\big(\gamma_1(t_1)\big)$ and $f'\big(\gamma_2(t_2)\big)$ are identical.

Thus, we conclude that:

$$\arg\big(f \circ \gamma_2'(t_2)\big) - \arg\big(f \circ \gamma_1'(t_1)\big) = \arg\big(\gamma_2'(t_2)\big) - \arg\big(\gamma_1'(t_1)\big),$$

which proves that f preserves the angle between the paths. Therefore, we have shown that f is conformal.

Thus, we conclude that Mobius transformations are indeed conformal maps because they are analytic functions. Now, we are ready to explore the geodesics of the upper half-plane with respect to the hyperbolic metric.

First, let's recall the hyperbolic metric on this upper half-plane. The square of the line element is given by:

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

This expression represents the hyperbolic metric defined on the upper half-plane, H^2.

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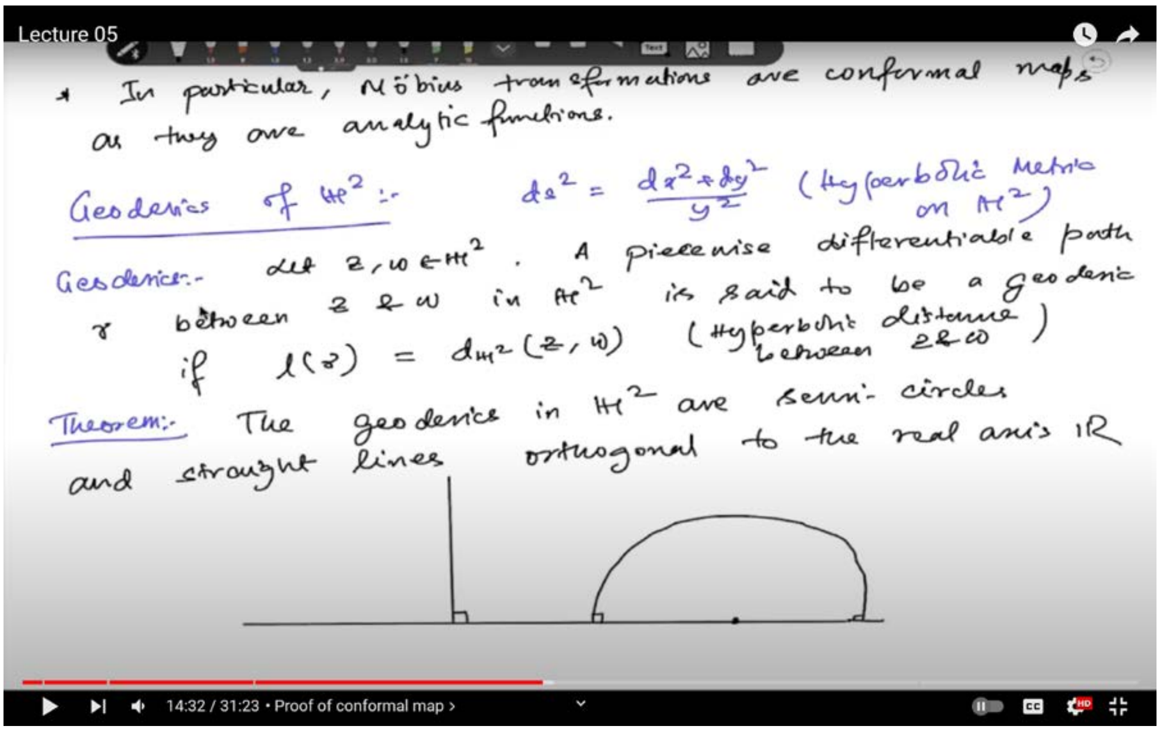

Now, let's define what we mean by geodesics. Consider two points z and w belonging to H^2. A piecewise differentiable path γ connecting z and w in H^2 is said to be a geodesic if the length of γ equals the hyperbolic distance $d_{H^2}(z, w)$ between these two points. In other words, if γ realizes the distance between z and w, we refer to γ as a geodesic.

Theorem:

The geodesics in the upper half-plane are semicircles and straight lines that are orthogonal to the real axis.

Let me illustrate this with a diagram. Here, we have the real line represented as the horizontal axis. If we take any vertical line, which is a straight line orthogonal to the real axis, this vertical line will serve as a geodesic.

Additionally, if we consider a semicircle that is centered on the real axis, it also qualifies as a geodesic, as it intersects the real axis at right angles. This geometric relationship is crucial for understanding the structure of geodesics within the framework of hyperbolic geometry.

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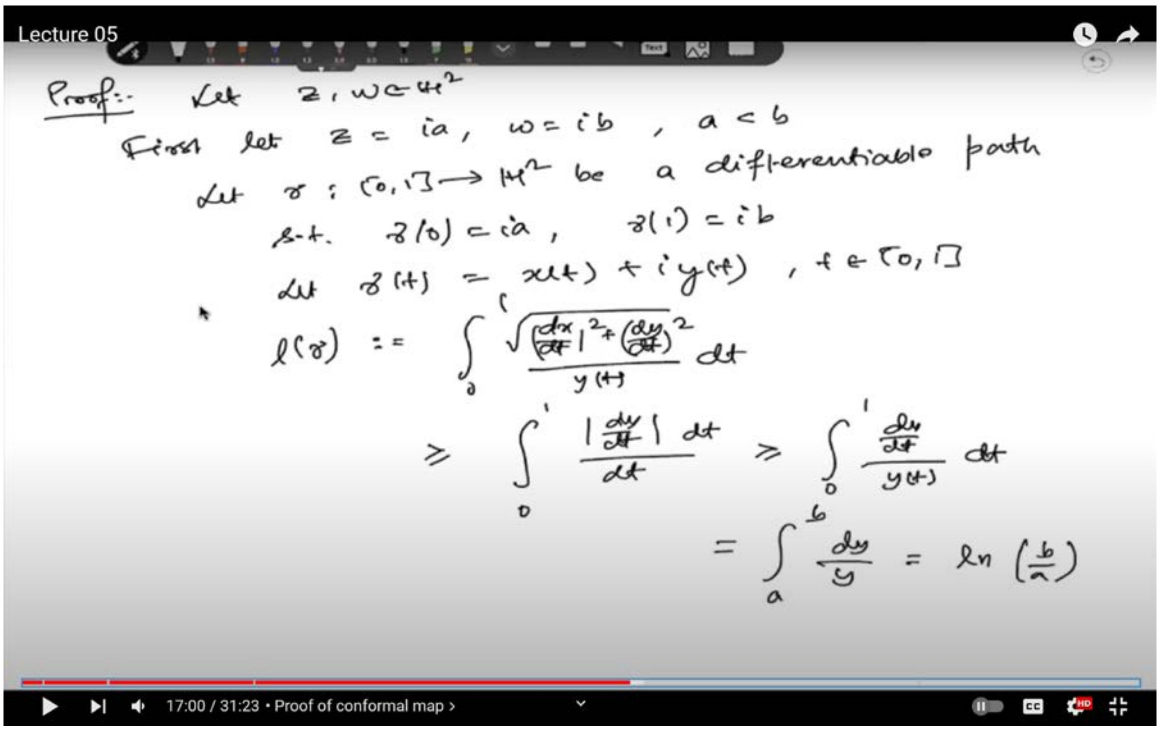

These are the geodesics of the upper half-plane. Now, let's move on to prove this theorem rigorously.

Let's consider two points, z and w, both belonging to the upper half-plane H^2. First, let z = i a and w = i b, where a < b. Let γ be a differentiable path joining these two points, z and w, such that γ(0) = i a and γ(1) = i b.

We can write γ(t) as γ(t) = x(t) + i y(t), where t varies within the closed interval [0, 1].

Now, according to the definition of the length of a curve in the hyperbolic metric, the length of γ is given by the following integral:

$$L(\gamma) = \int_0^1 \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y(t)}\, dt.$$

Notice that in this integrand, the numerator is always greater than or equal to $\left|\frac{dy}{dt}\right|$, which leads to the inequality:

$$L(\gamma) \geq \int_0^1 \frac{\left|\frac{dy}{dt}\right|}{y(t)}\, dt.$$

Since $\left|\frac{dy}{dt}\right|$ is positive, we can simplify further, obtaining:

$$L(\gamma) \geq \int_0^1 \frac{dy}{y(t)}\, dt.$$

This integral simplifies to:

$$L(\gamma) \geq \int_a^b \frac{dy}{y} = \log\left(\frac{b}{a}\right).$$

Therefore, we have shown that the length of the path γ is greater than or equal to $\log\left(\frac{b}{a}\right)$.

Next, let's consider a specific path β(t), defined as:

$$\beta(t) = i\big(a + t(b - a)\big),$$

where t again varies in the closed interval [0, 1]. The image of β is a straight line joining i a and i b.

For this path β(t), we compute the length. Since the path is vertical and straightforward, its length precisely equals the hyperbolic distance between the points i a and i b, which is $\log\left(\frac{b}{a}\right)$.

Thus, the path β, which is a vertical line, realizes the minimal distance between i a and i b, confirming that vertical lines are geodesics in the hyperbolic upper half-plane.

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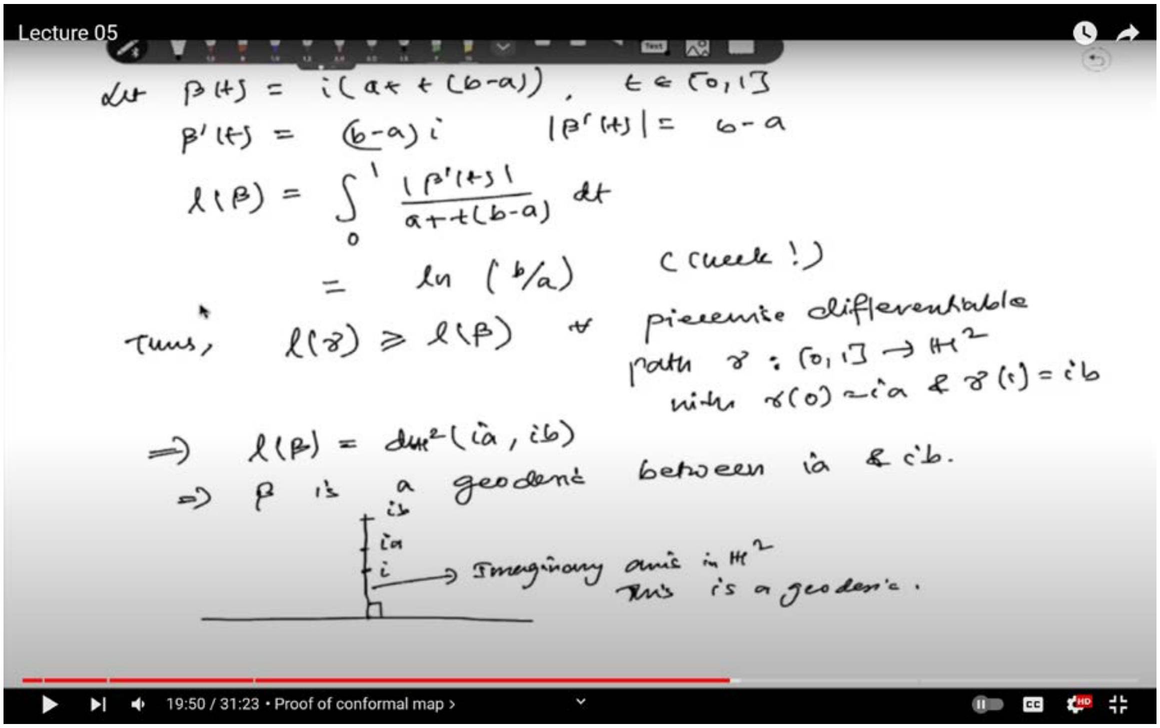

Let’s consider the convex combination of the points i a and i b. Specifically, we define the path $\beta(t) = i\big(a + t(b-a)\big)$, where $t \in [0,1]$. The derivative of this path, β'(t), becomes:

$$\beta'(t) = i(b-a).$$

Now, the modulus of β'(t), denoted as |β'(t)|, is simply:

$$|\beta'(t)| = b - a.$$

Next, let's compute the length of β. The length of β in the hyperbolic metric is given by:

$$L(\beta) = \int_0^1 \frac{|\beta'(t)|}{\mathrm{Im}\big(\beta(t)\big)}\, dt.$$

Substituting the expression for β(t), where β(t) = i(a + t(b - a)), we get:

$$L(\beta) = \int_0^1 \frac{b-a}{a + t(b-a)}\, dt.$$

This integral simplifies to:

$$L(\beta) = \log\left(\frac{b}{a}\right),$$

which you can verify. Thus, we have shown that the length of the path β equals $\log\left(\frac{b}{a}\right)$.

Now, since we previously established that the length of any piecewise differentiable path γ is always greater than or equal to $\log\left(\frac{b}{a}\right)$, it follows that:

$$\text{Length of } \gamma \geq \text{Length of } \beta.$$

This holds for all piecewise differentiable paths γ joining i a and i b. Therefore, the length of β is indeed the hyperbolic distance between i a and i b. As a result, the path β is a geodesic in the upper half-plane.

Consequently, we have proven that the vertical line along the imaginary axis in the upper half-plane is a geodesic.

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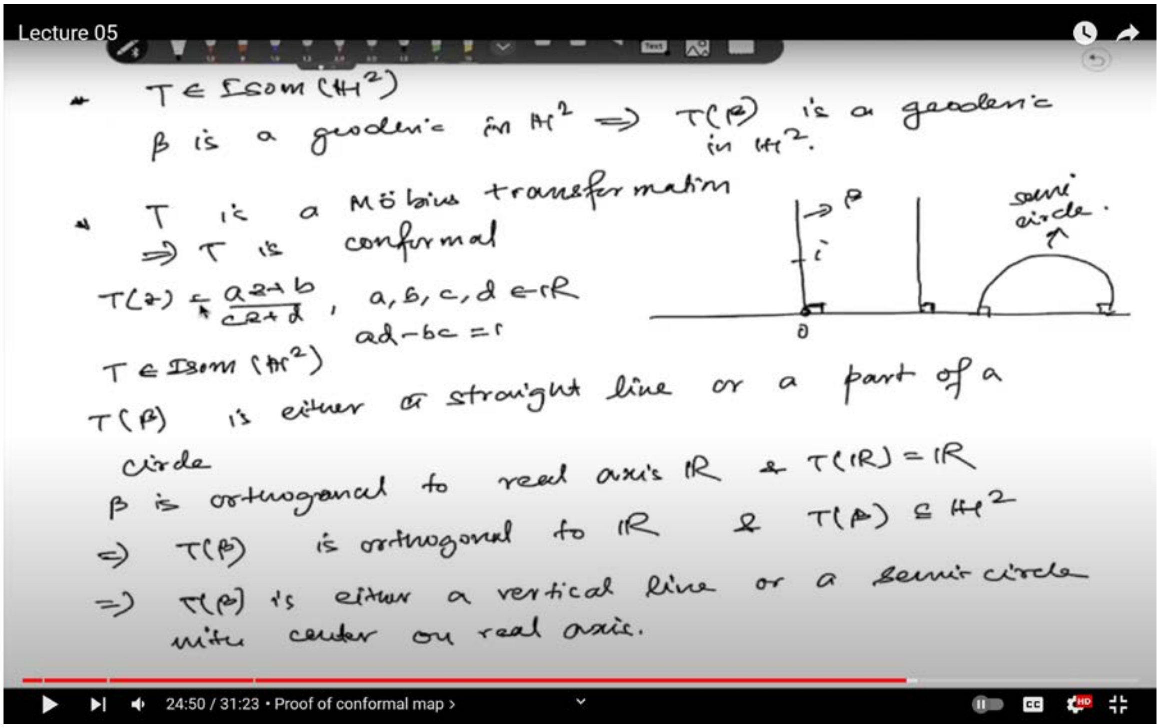

Now, let's explore how to find the other geodesics in the upper half-plane. The key tool here is the concept of isometries. Consider an isometry T, which is a transformation that preserves distances in the upper half-plane H^2. Because isometries preserve distances, if β is a geodesic in H^2, then the image of β under T, denoted T(β), will also be a geodesic in H^2.

Next, let T be a Möbius transformation. Since Möbius transformations are conformal, they preserve angles. Suppose we start with the geodesic β, which we've already identified as the vertical imaginary axis. Since β is a geodesic, T(β) must also be a geodesic. When T is a Möbius transformation, T(β) will either be a straight line or a part of a circle. This follows from a theorem we proved in the previous class, which states that Möbius transformations take straight lines and circles to other straight lines or circles.

Now, let's consider T to be an element of the group PSL(2, R), which represents transformations of the form:

$$T(z) = \frac{Az + B}{Cz + D},$$

where A, B, C, D are real numbers, and AD - BC = 1 (the determinant condition that ensures T is in PSL(2, R)). We know that such transformations are isometries of the upper half-plane, and we've also established that they are Möbius transformations. Therefore, T(β), the image of β, will be either a straight line or part of a circle, just as before.

However, since T is conformal, it must preserve the angles. This means that if T(β) is a straight line, it must remain orthogonal to the real axis. So, the only straight lines that can be geodesics are vertical lines, which are orthogonal to the real axis. If T(β) is a part of a circle, the same orthogonality condition applies, it must be orthogonal to the real axis. Therefore, the image of β cannot be just any circle, but specifically a semicircle centered on the real axis.

Thus, we've proven that the only geodesics in the upper half-plane are either vertical lines or semicircles that are orthogonal to the real axis. Additionally, since T(R) = R, this implies that T(β), whether a vertical line or a semicircle, is contained entirely within the upper half-plane. Consequently, all geodesics in the upper half-plane are either vertical lines or semicircles with their centers on the real axis.

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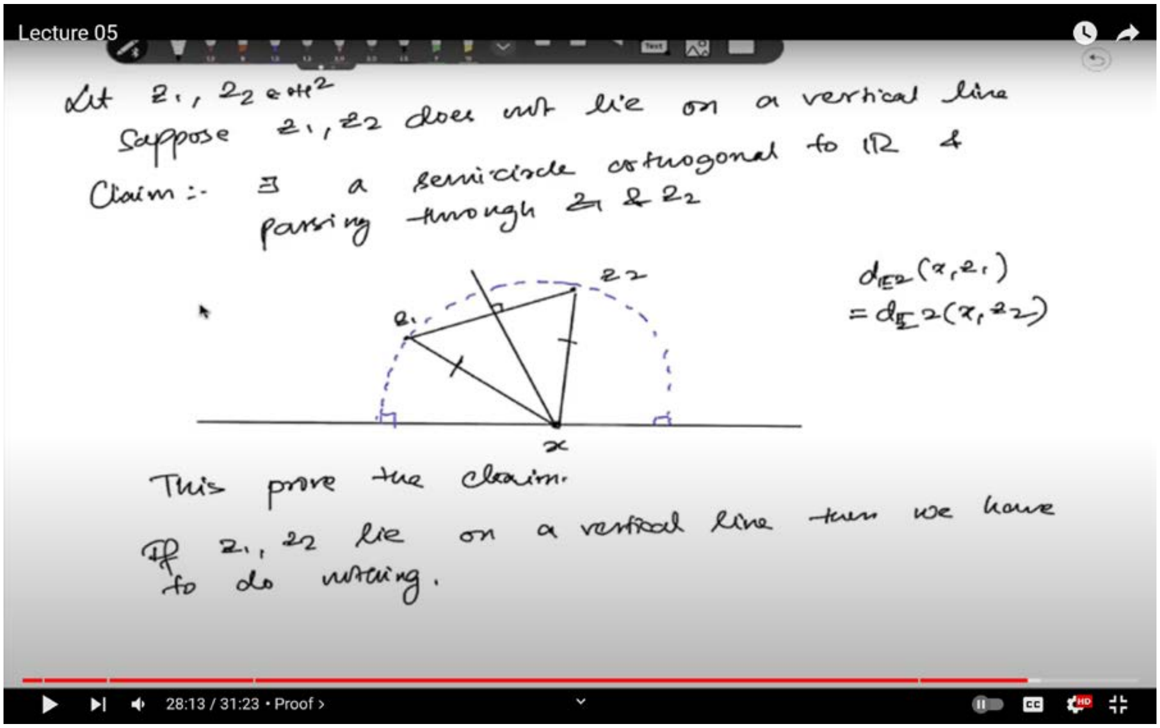

What we have established is that vertical lines and semicircles orthogonal to the real axis are indeed the geodesics in the upper half-plane. But an important question remains: are there any other types of geodesics? The answer, as we will demonstrate, is no, there are no other geodesics beyond these.

Now, suppose we are given two points, z_1 and z_2, both of which lie in the upper half-plane H^2. Let's consider the case where z_1 and z_2 do not lie on a vertical line. The claim is that there exists a semicircle, orthogonal to the real axis, that passes through both z_1 and z_2.

How do we prove this? Begin by considering the real axis and the two points z_1 and z_2. First, draw a straight line segment joining z_1 and z_2. Next, find the perpendicular bisector of this line segment. The perpendicular bisector will intersect the real axis at some point, let's call this point x.

Now, join z_1 to x with a straight line, and similarly join z_2 to x with another straight line. By construction, the Euclidean distance from x to z_1 will be equal to the Euclidean distance from x to z_2. Finally, we draw a semicircle with its center at x and a radius equal to the distance between x and z_1 (which is also the distance from x to z_2). This semicircle, by definition, is orthogonal to the real axis and passes through both z_1 and z_2.

Thus, we have proven the claim: if two points z_1 and z_2 do not lie on a vertical line, there exists a unique semicircle, orthogonal to the real axis, that connects them. On the other hand, if z_1 and z_2 do lie on a vertical line, then the geodesic connecting them is simply the vertical line itself, and nothing more needs to be done.

Therefore, we conclude that no other types of geodesics exist in the upper half-plane. If the points lie on a vertical line, the geodesic is that line. If the points do not lie on a vertical line, the geodesic is a semicircle orthogonal to the real axis. This completes the proof that these are the only geodesics in the upper half-plane.

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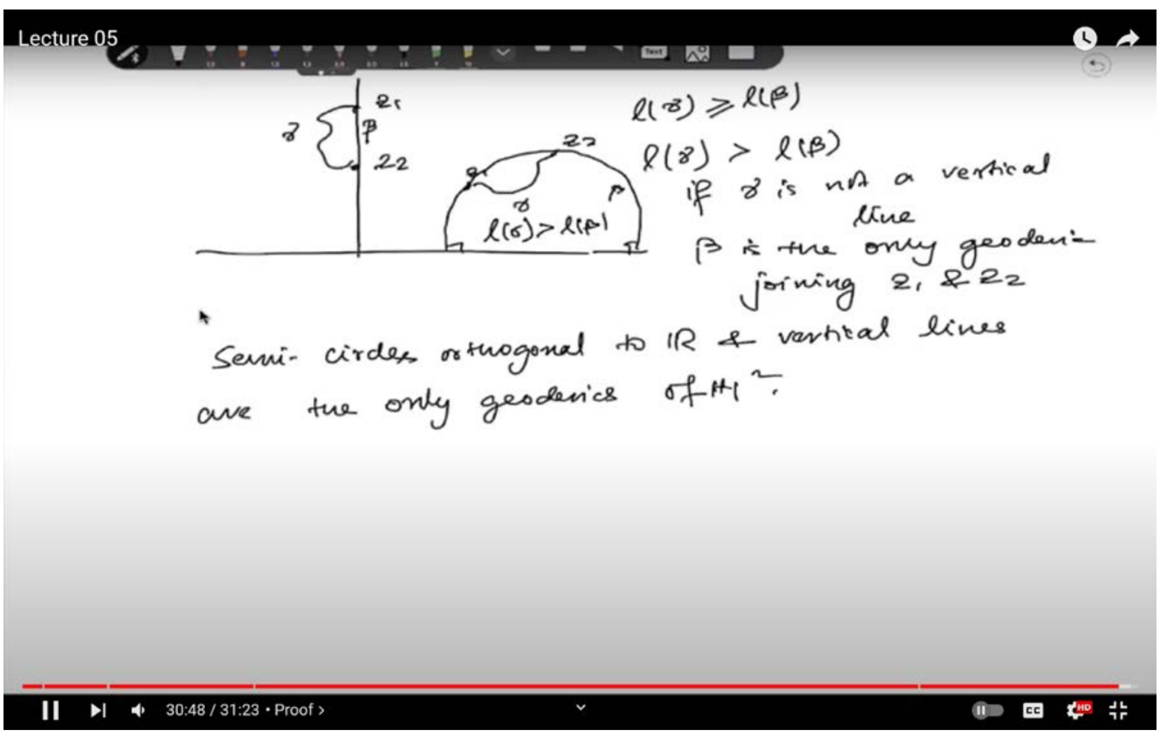

Let’s consider two points, z_1 and z_2, starting with the case where both points lie on a vertical line. Suppose we take any other path, let's call it γ. We have already established the following: if β is the vertical geodesic joining z_1 and z_2, and γ is any differentiable path connecting the same two points, the length of γ will always be greater than or equal to the length of β.

If γ is not a vertical line, then the length of γ will strictly exceed that of β. This confirms that β is the only geodesic connecting z_1 and z_2 when these two points lie on a vertical line.

Now, if z_1 and z_2 do not lie on a vertical line, there will always exist a semicircle with its center on the real axis that passes through both points. Applying a similar argument, we can show that if we take any differentiable or piecewise differentiable path γ joining z_1 and z_2, and the image of γ does not coincide with that of the semicircular geodesic β, the length of γ will be strictly greater than the length of β.

Thus, we have conclusively proven that semicircles orthogonal to the real axis and vertical lines are the only geodesics in the upper half-plane. We can stop here with the proof.