

# An Introduction to Hyperbolic Geometry

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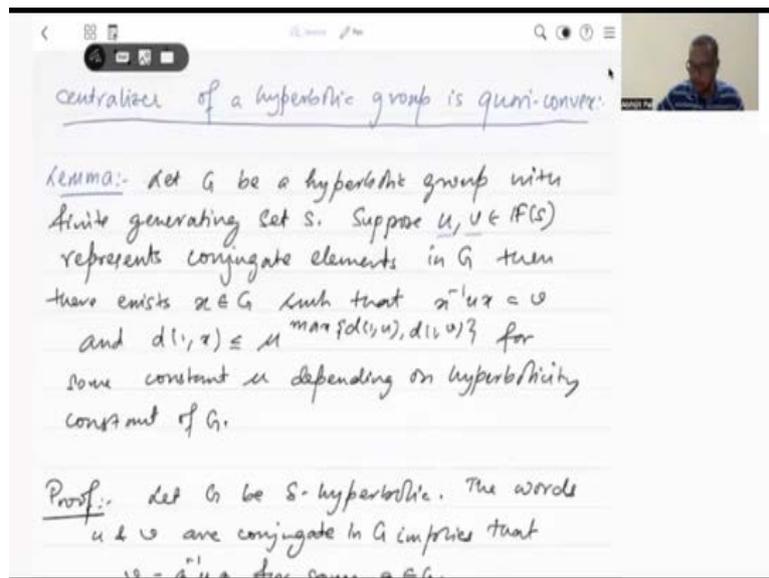
Department of Mathematics & Statistics

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Lecture – 40

Understanding Quasi-Convex Subgroups: The Centralizer of Hyperbolic Groups

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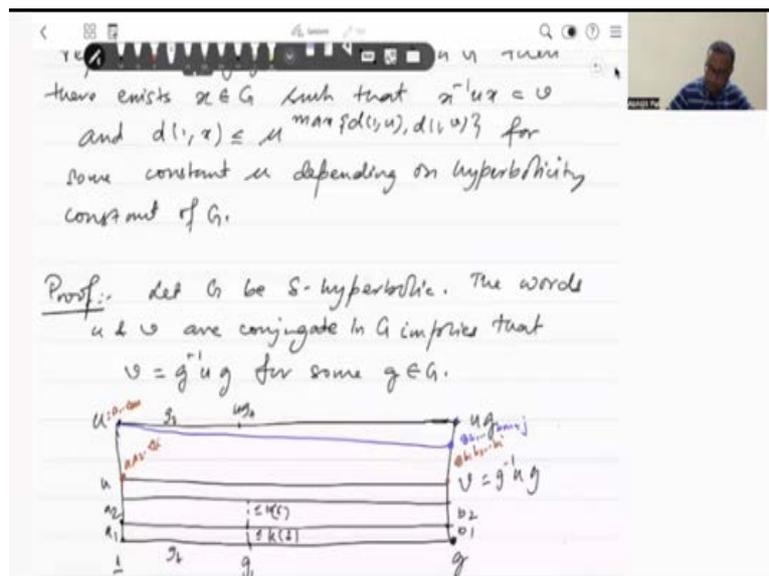
Hello everyone! In our last lecture, we explored the concept of quasi-convex subgroups. Today, we will prove that the centralizer of a hyperbolic group is indeed a quasi-convex subgroup.

Let's begin with a lemma that states the following: Let  $G$  be a hyperbolic group, which implies that it is a finitely generated group. We denote  $S$  as a finite generating set for this group  $G$ . Now, consider two words  $u$  and  $v$  in the free group generated by  $S$ .

Assuming that the elements  $u$  and  $v$  represent conjugate elements in the group  $G$ , we can claim that there exists an element  $x$  in this group  $G$  such that  $x^{-1}ux = v$ . Furthermore, the distance between the identity element  $1$  and  $x$  in the group  $G$  is less than or equal to  $\mu$  raised to the power of the maximum of the distances between  $1$  and  $u$ , and between  $1$  and  $v$ . Here, the constant  $\mu$  depends solely on the hyperbolicity constant of the group  $G$ .

To summarize, we start with a hyperbolic group  $G$  that has a finite generating set  $S$ . We take two elements  $u$  and  $v$  in the free group generated by  $S$ , which also represent elements in the group  $G$ . Assuming they are conjugate in  $G$ , we can assert the existence of an element  $x$  such that  $x^{-1}ux = v$  and that the distance from  $1$  to  $x$  is less than or equal to  $\mu$  raised to the maximum of the distances from  $1$  to  $u$  and from  $1$  to  $v$ . The distances are measured in the group  $G$ , and the constant  $\mu$  is based only on the hyperbolicity of  $G$ .

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What is the proof of this lemma? To establish it, we will rely on the exercise we discussed in the previous class. Let  $G$  be a  $\delta$ -hyperbolic group. This means that the Cayley graph of  $G$  with respect to the generating set  $S$  forms a hyperbolic metric space, specifically a  $\delta$ -hyperbolic metric space.

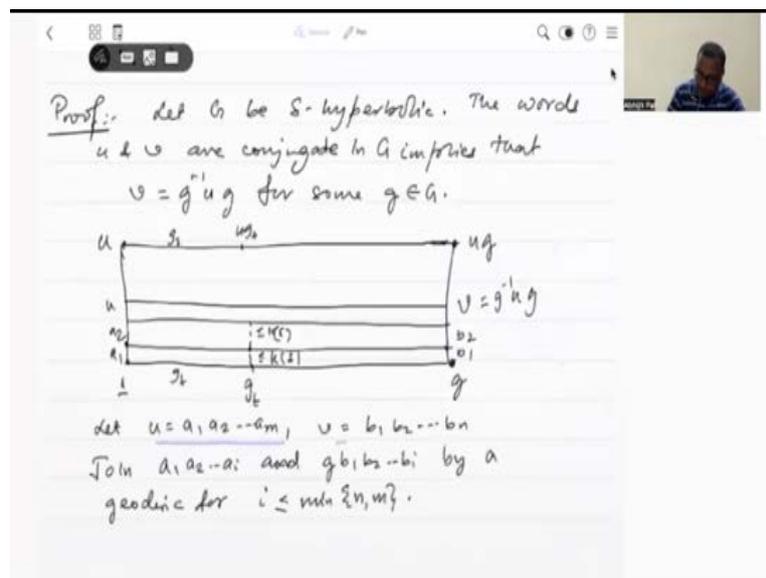
Now, consider the two words  $u$  and  $v$ , which are conjugate in  $G$ . This implies that there exists some element  $g$  in the group  $G$  such that  $v = g^{-1}ug$ .

To visualize this, let's consider the identity element  $1$  and the element  $u$ . We can connect  $1$  to  $u$  via a geodesic in the Cayley graph. Next, we take the element  $g$  and the element  $u g$ , and connect  $g$  and  $u g$  by another geodesic in the group  $G$ .

The word representing this geodesic will be precisely  $g^{-1} u g$ , which is equivalent to  $v$ . Therefore,  $v$  corresponds to the geodesic that joins  $g$  and  $u g$  in the Cayley graph of the

group  $G$ .

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Now, let us define  $u$  as  $a_1, a_2, \dots, a_m$  and  $v$  as  $b_1, b_2, \dots, b_n$ , where the  $a_i$ 's and  $b_i$ 's belong to the set  $S \cup S^{-1}$ . Here,  $S$  represents the generating set of the group  $G$ . Consequently, if we take the geodesic from the identity element  $1$  to  $u$ , the first segment will be labeled by  $a_1$ , the second by  $a_2$ , and so forth.

At the other end, the geodesic connecting  $g$  to  $ug$  will also have segments labeled in a similar manner. The first segment will be labeled by  $b_1$ , and the second by  $b_2$ , and so on.

Now, let us connect the points  $a_1, a_2, \dots, a_i$  and  $g b_1, b_2, \dots, b_i$  by a geodesic for  $i$  less than or equal to the minimum of  $n$  and  $m$ . Suppose the sequence is represented as  $a_1, a_2, \dots, a_i$  on one side and  $g b_1, b_2, \dots, b_i$  on the other. The distance from  $1$  to the point  $a_i$  will correspond to  $i$ , and similarly, the distance from  $g$  to the point  $g b_i$  will also equal  $i$ .

It's important to note that since  $u = a_1, a_2, \dots, a_m$  and  $v = b_1, b_2, \dots, b_n$ , it follows that  $i$  will be less than or equal to the minimum of  $n$  and  $m$ .

Now, let us assume that  $m$  is less than or equal to  $n$ . In this case, we will connect the points  $a_1, a_m$  to  $g b_1, b_{m+j}$  by a geodesic. Imagine we have a setup like this, where the path from  $g b_1$  to  $g b_m$  is clearly defined. We will extend this connection to include all

segments leading to  $g_{b_{m+j}}$ .

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Suppose  $m \leq n$ . Join  $u = a_1 \dots a_m$  to  $g_{b_1 \dots b_{m+j}}$  by a geodesic.

Let  $g_t \in [1, g]$  be such that  $d(1, g_t) = t$ .

Then  $d(u, u g_t) = t$ .

Let  $k(s) = 4s + 3$ .

By the exercise,  $d(g_s, g_{s'}) \leq k(s)$  where  $g_s \in [a_1, g_{b_1}]$  &  $d(a_1, g_s) = s$ .

We continue this process for all geodesics  $[a_1, a_i, g_{b_1}, \dots, g_{b_i}]$  &  $[u, g_{b_1}, \dots, g_{b_{m+j}}]$

Let  $n = k(s) \max\{d(1, u), d(1, v)\}$

Next, let us consider a vertex  $g_t$  located on the geodesic that connects 1 to  $g$ , ensuring that the distance between 1 and  $g_t$  is equal to  $t$ . Consequently, the distance between  $u$  and  $u g_t$  will also equal  $t$ . Thus, if we select any point  $g_t$  along this geodesic from 1 to  $g$ , we observe that the distance from 1 to  $g_t$  is identical to the distance from  $u$  to  $u g_t$ .

Now, let's take a moment to note this crucial observation.

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Let  $n = k(s) \max\{d(1, u), d(1, v)\}$

Then  $d(g_s, u g_s) \leq k(s) \max\{d(1, u), d(1, v)\}$

$\Rightarrow d(1, g_s u g_s) \leq n$ .

Let  $B(n) =$  set of all words of length  $\leq n$ .

Construct a finite graph  $G(n)$  with vertex set as  $B(n)$ . Two vertices  $v_1, v_2 \in B(n)$  have an edge between them if they differ by at most one letter.

Now, let us apply the exercise we discussed in the last class. We know that the distance between  $g_t$  and  $g_{t'}$  is less than or equal to  $k\delta$ , where  $g_{t'}$  lies on the geodesic connecting  $a_1$  and  $g_{b_1}$ . Here, the distance between  $a_1$  and  $g_{t'}$  is equal to  $t$ . Therefore, if we select a point  $g_{t'}$  in this context, with  $a_1$  as the vertex, the distance between  $a_1$  and  $g_{t'}$  is precisely  $t$ , which is also the same as the distance between  $1$  and  $g_t$ .

Considering these two geodesics, the one connecting  $1$  to  $g$  and the one joining  $a_1$  to  $g_{b_1}$ , the distance between  $g_t$  and  $g_{t'}$  must be less than or equal to  $k\delta$ . This  $k\delta$  arises from the earlier exercise, where we found that  $k\delta = 4\delta + 3$ . Thus, it is clear that the distance between  $g_t$  and  $g_{t'}$  is indeed less than or equal to  $k\delta$ .

We will now continue this process for all geodesics connecting  $a_1, a_2, \dots, a_i$  to  $g_{b_1}, b_2, \dots, b_i$ , as well as for the geodesic connecting  $u$  to  $g_{b_{1+m}}$ . We can define  $n$  as  $k\delta$  multiplied by the maximum distance between  $1$  and  $u$ , and between  $1$  and  $b$ .

Consequently, we have established that the distance between  $g_t$  and  $u g_t$  is less than or equal to  $k\delta$  times the maximum of the distances between  $1$  to  $u$  and  $1$  to  $b$ . To visualize this, if we take  $g_t$  and relate it to  $u g_t$ , the distance between these two points will remain less than or equal to  $n$ , which is effectively  $k\delta$  added to  $k\delta$  repeatedly. Thus, this leads us to conclude that the distance between  $g_t$  and  $u g_t$  is indeed less than or equal to  $n k\delta$ , where  $n$  is less than or equal to the maximum distance between  $1$  to  $u$  and the distance between  $g$  and  $u g$ , which corresponds to the distance between  $1$  and  $b$ .

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Let  $n = k(\delta) \max\{d(1, u), d(1, b)\}$

Then  $d(1, u g_t) \leq k(\delta) \max\{d(1, u), d(1, b)\}$

$\Rightarrow d(1, g_t^{-1} u g_t) \leq n$ .

Let  $B(n) =$  set of all words of length  $\leq n$ .

Construct a finite graph  $G(n)$  with vertex set as  $B(n)$ . Two vertices  $v_1, v_2 \in B(n)$  have an edge if  $v_2 = a^{-1} v_1 a$  for some  $a \in S$ .  
( $a$  is a generator of  $G_0$ )

$u, v \in B(n)$ .  $u \sim v$  are conjugate if and only if  $u, v$  lie in the same path component of  $G(n)$ .

Thus, the distance between  $g_t$  and  $u g_t$  can be expressed as being less than or equal to  $k\delta + k\delta$  repeatedly, leading us to conclude that this distance is less than or equal to  $n k\delta$ , where  $n$  is indeed constrained by the maximum distance from  $1$  to  $u$  and the distance from  $g$  to  $u g$ , ultimately equating to the distance between  $1$  and  $b$ .

Yes, let's clarify our findings. The distance between  $g_t$  and  $u g_t$  is indeed less than or equal to  $n$ . Consequently, we can say that the distance from  $1$  to the conjugate  $g_t^{-1} u g_t$  is also less than or equal to  $n$ . Here, we have defined  $n$  as  $k\delta$  multiplied by the maximum of the distances from  $1$  to  $u$  and from  $1$  to  $b$ .

Now, what we aim to achieve is to find an element  $x$  such that the distance from  $1$  to  $x$  is less than or equal to  $\mu$  times the maximum of the distances between  $1$  to  $u$  and  $1$  to  $b$ . Our objective is to ensure that the length of  $x$  is uniformly bounded in relation to the hyperbolic constant of  $G$  and the distances from  $1$  to  $u$  and from  $1$  to  $b$ . So how do we accomplish this?

To obtain such an  $x$ , we will construct another finite graph, which will be defined as follows. Let us denote  $B_n$  as the set of all words with lengths less than or equal to  $n$ . Next, we will construct a finite graph  $G_n$  with the vertex set being  $B_n$ . In this graph, two vertices  $v_1$  and  $v_2$  are connected by an edge if  $v_2$  can be expressed as  $a^{-1} v_1 a$  for some generator  $a$  belonging to  $S$  or  $S^{-1}$ , where  $S$  is the generating set of the group  $G$ .

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$G_n$  has  $(2|S|)^n$  vertices.  
 Any injective path has length at most  $(2|S|)^n$ .  
 Thus,  $u$  &  $v$  are conjugate in  $G$  implies there exists a word  $w$  of length  $\leq (2|S|)^n$   
 where  $n = k(S) \max\{d(1, u), d(1, v)\}$   
 Let  $\mu = (2|S|)^{k(S)}$ , then we have the required result.  
Theorem: Let  $G$  be a hyperbolic group. The centralizer  $C(g)$  of any element  $g \in G$



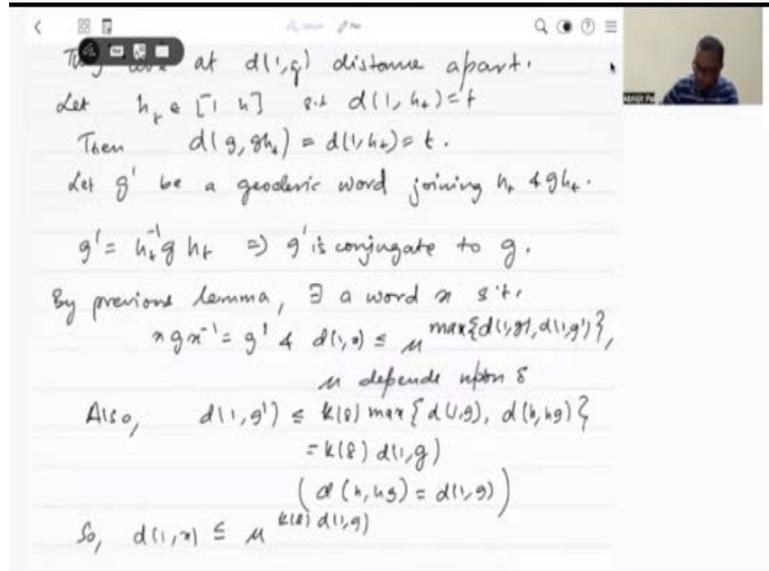
Let us begin our discussion with a hyperbolic group  $G$ . We will denote the centralizer of an element  $g$  as  $C_G(g)$ . Our goal is to demonstrate that this centralizer is quasi-convex, with the quasi-convexity constant depending solely on the length of  $g$  and the hyperbolicity constant of the group  $G$ .

Now, let's take an element  $h$  from the centralizer of  $g$ . Since  $h$  belongs to the centralizer, we have the relation  $ghg^{-1} = h$ .

To visualize this, let's consider the following setup. First, we connect the identity element  $1$  to  $h$  using a geodesic, and we also connect  $1$  to  $g$  with another geodesic. Additionally, we can join  $h$  to  $gh$  by yet another geodesic. It is essential to note that  $gh$  is equal to  $hg$ .

As a result, the length of the geodesic connecting  $1$  to  $h$  will be the same as that of the geodesic connecting  $1$  to  $g$ . Now, let's focus on the geodesics  $1h$  and  $ggh$  and examine these two geodesics more closely.

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The two geodesics are indeed at a distance of 1 from each other, which has significant implications. Specifically, if we consider the endpoints of these two geodesics, one connecting  $1$  to  $h$  and the other connecting  $1$  to  $g$ , we find that the distance between the endpoints  $h$  and  $gh$  (which is also equal to the distance between  $h$  and  $hg$ ) is the same as the distance between  $1$  and  $g$ . Thus, we can conclude that the two geodesics are

separated by a distance of 1.

Next, let's introduce a point  $h_t$  on the geodesic joining 1 to  $h$  such that the distance between 1 and  $h_t$  is equal to  $t$ . If we then consider the point  $gh_t$ , the distance between  $g$  and  $gh_t$  is equal to the distance between 1 and  $h_t$ , which is  $t$ . Hence,  $h_t$  is positioned on the geodesic, and this leads us to the conclusion that the distance between  $g$  and  $gh_t$  is indeed equal to the distance from 1 to  $h_t$ .

Now, if we connect  $h_t$  and  $gh_t$  by a geodesic, let's denote this geodesic as  $g'$ . We can express  $g'$  as  $h_t^{-1}gh_t$ , signifying that  $g'$  represents the geodesic joining  $h_t$  and  $gh_t$ . This implies that  $g'$  is conjugate to  $g$ .

According to the previous lemma, there exists a word  $x$  such that  $xgx^{-1} = g'$ , and the distance between 1 and  $x$  is less than or equal to  $\mu$  raised to the maximum of the distances between 1 and  $g$ , as well as 1 and  $g'$ . Importantly, this constant  $\mu$  depends solely on the hyperbolicity constant  $\delta$ .

Furthermore, it's essential to note that the distance between 1 and  $g'$  is less than or equal to  $k\delta$  multiplied by the maximum of the distances between 1 and  $g$ , as well as the distance between  $h$  and  $gh$ . This conclusion also stems from the previous lemma. Consequently, the distance between  $h_t$  and  $gh_t$  will be less than or equal to  $k\delta$  times the maximum of the distances between 1 and  $g$ , along with the distance from  $h$  to  $gh$ .

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$\mu = \mu$   $d(1, g) \leq \mu$   
 $\mu$  depends upon  $\delta$   
 Also,  $d(1, g') \leq k(\delta) \max\{d(1, g), d(h, hg)\}$   
 $= k(\delta) d(1, g)$   
 $(d(h, hg) = d(1, g))$   
 So,  $d(1, x) \leq \mu^{k(\delta) d(1, g)}$   
 Claim:  $h_t x \in \langle g \rangle$   
 $(h_t x)g = h_t xg = h_t g'x$   
 $= g(h_t x)$   
 $\Rightarrow h_t x \in \langle g \rangle$   
 $d(h_t, h_t x) \leq \mu^{k(\delta) d(1, g)}$   
 This is true for all  $h_t \in [1, h]$ ,  $h \in \langle g \rangle$   
 So,  $\langle g \rangle$  is quasi-convex.

Moreover, the distance from  $h_t$  to  $gh_t$  is essentially equal to the distance from 1 to  $g'$ , which we have previously established. This brings us to a clearer understanding of the relationships among these elements and distances within the group.

The distance from 1 to  $g'$  is indeed less than or equal to  $k\delta$  times the maximum of the two distances we've discussed. Furthermore, the distance from 1 to  $g$  is the same as the distance between  $h$  and  $hg$ . Therefore, we can express this as  $k\delta$  times the distance from 1 to  $g$ .

From this inequality, we can deduce that the distance between 1 and  $x$  is less than or equal to  $\mu^{k\delta}$  times the distance from 1 to  $g$ . Our goal is to prove that the centralizer is quasi-convex.

We have identified the point  $h_t$ , and we need to find a point in the centralizer such that the distance from  $h_t$  to this point is less than or equal to a function of  $\delta$  and the length of  $g$ . We will demonstrate that the product  $h_t x$  actually lies in the centralizer. Importantly, we know that the length of  $x$  is bounded in relation to the hyperbolicity constant and the length of  $g$ .

The claim we wish to establish is that  $h_t x$  belongs to the centralizer of the element  $g$ . We can express this relationship as follows:  $h_t x g$  is equal to  $h_t g' x$ . This equality holds because we know  $x g$  is equivalent to  $g' x$ . Therefore, substituting this back in gives us:

$$h_t x g = h_t g' x$$

Additionally, we have  $h_t g' = g h_t$  since  $g'$  can be represented as  $h_t^{-1} g h_t$ . Consequently, we arrive at the conclusion that:

$$h_t x g = g h_t x$$

This establishes that  $h_t x$  indeed belongs to the centralizer.

Moreover, the distance between  $h_t$  and  $h_t x$  is less than or equal to  $\mu^{k\delta}$  times the distance between 1 and  $g$ . This holds true for all  $h_t$  located on the geodesic joining 1 to  $h$ , with  $h$  itself belonging to the centralizer. Thus, we can conclude that the centralizer is a quasi-convex subgroup. We will pause our discussion here, and in the next class, we will prove that the intersection of two quasi-convex subgroups is, once again, a quasi-

convex subgroup.