An Introduction to Hyperbolic Geometry

Prof. Abhijit Pal

Department of Mathematics and Statistics

Indian Institute of Technology - Kanpur

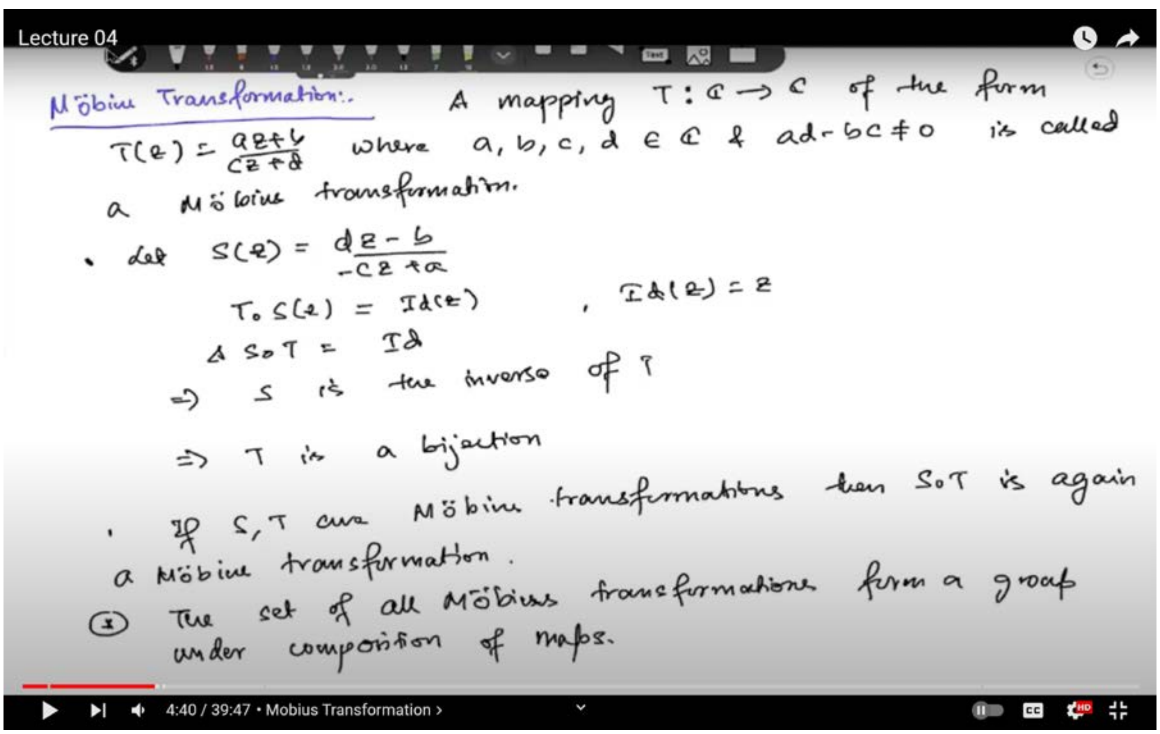
Module - 2

Lecture - 4

Exploring Mobius Transformations and the Geodesics of the Upper Half-Plane

In the previous lecture, we discussed that the group PSL(2, $\mathbb{R}$) is a subgroup of the isometry group of the upper half-plane. Building upon that, we will now prove that geodesics in the upper half-plane, with respect to the hyperbolic metric, are represented by semicircles whose centers lie on the real axis, as well as by vertical lines. To establish this, we need to explore Möbius transformations.

(Refer Slide Time: 04:40)



Let us first delve into Möbius transformations by clearly defining what they entail. A Möbius transformation is a mapping, denoted by T, from the complex plane to itself, given by the formula:

$$T(z) = \frac{az + b}{cz + d}$$

where a, b, c, and d are complex numbers, and the transformation is valid provided $ad - bc \neq 0$.

So, where this a, b, c, d are my complex numbers and this ad minus bc is not equal to 0. This mapping is referred to as a Möbius transformation. One of the key results we will demonstrate in this lecture is that Möbius transformations map circles and straight lines to either circles or straight lines. But before diving into that, let us first establish why Möbius transformations are bijections.

Consider the following transformation: let S(z) be defined as

$$S(z) = \frac{dz - b}{-cz + a}$$

This S is also a Möbius transformation since the condition $ad - bc \neq 0$ holds. Therefore, S is valid as a Möbius transformation. Now, if we take the composition of T and S, denoted as $T \circ S$, we find that this is nothing but the identity map, i.e.,

$$(T \circ S)(z) = z$$

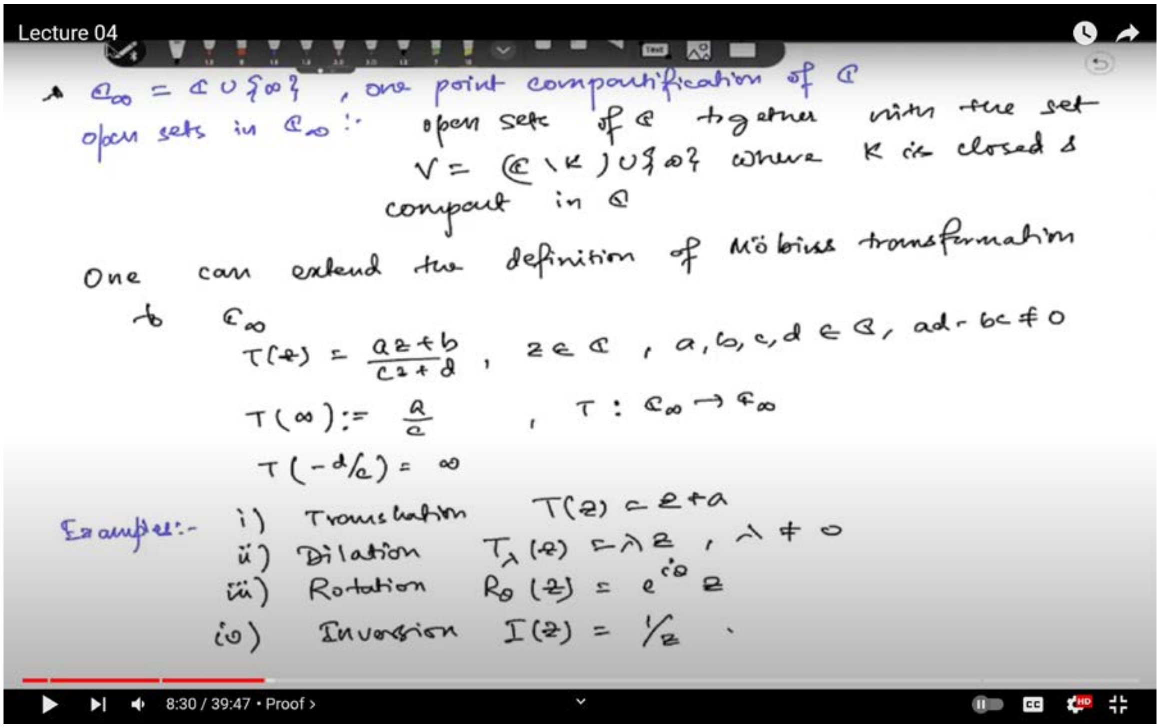
Similarly, if we compose S and T, denoted as $S \circ T$, we again get the identity map. This implies that S is the inverse of T, proving that T is invertible. Consequently, this shows that T is a bijection.

Furthermore, it is important to note that if S and T are both Möbius transformations, their composition $S \circ T$ is also a Möbius transformation. This result confirms that the set of all Möbius transformations forms a group under the operation of composition.

Lastly, this definition of Möbius transformations can be extended to $C \cup \{\infty\}$, where $C \cup \{\infty\}$ represents the one-point compactification of the complex plane C.

Let me first clarify the concept. When we write C_∞, it refers to $C \cup \{\infty\}$, which is the one-point compactification of the complex plane C. Now, what are the open sets in this one-point compactification? These are the open sets of C, along with the set $V = (C \setminus K) \cup \{\infty\}$, where K is a closed and compact subset of C.

(Refer Slide Time: 08:30)



These are the open sets in C_∞. With this, we can extend the definition of a Möbius transformation to C_∞, or $C \cup \{\infty\}$. So, how do we define this extension?

For a Möbius transformation T, we have:

$$T(z) = \frac{az + b}{cz + d}$$

for $z \in C$, where a, b, c, d are complex numbers, and $ad - bc \neq 0$. Additionally, the transformation at infinity is defined as:

$$T(\infty) = \frac{a}{c}$$

Thus, T is now a well-defined map fromC_∞ to C_∞. In fact, you can verify that:

$$T\left(-\frac{d}{c}\right) = \infty$$

Now, let's explore some specific examples of Möbius transformations.

1. Translation:

$$T(z) = z + a$$

where $a \in C$. This is a Möbius transformation.

2. Dilation:

$$T_{\lambda}(z) = \lambda z$$

where $\lambda \neq 0$, another example of a Möbius transformation.

3. Rotation:

$$R_{\theta}(z) = e^{i\theta} z$$

where θ is a real number, giving a rotation about the origin.

4. Inversion:

$$I(z) = \frac{1}{z}$$

Each of these transformations is a Möbius transformation, and it can be shown that all Möbius transformations are compositions of translations, dilations, rotations, and inversions.

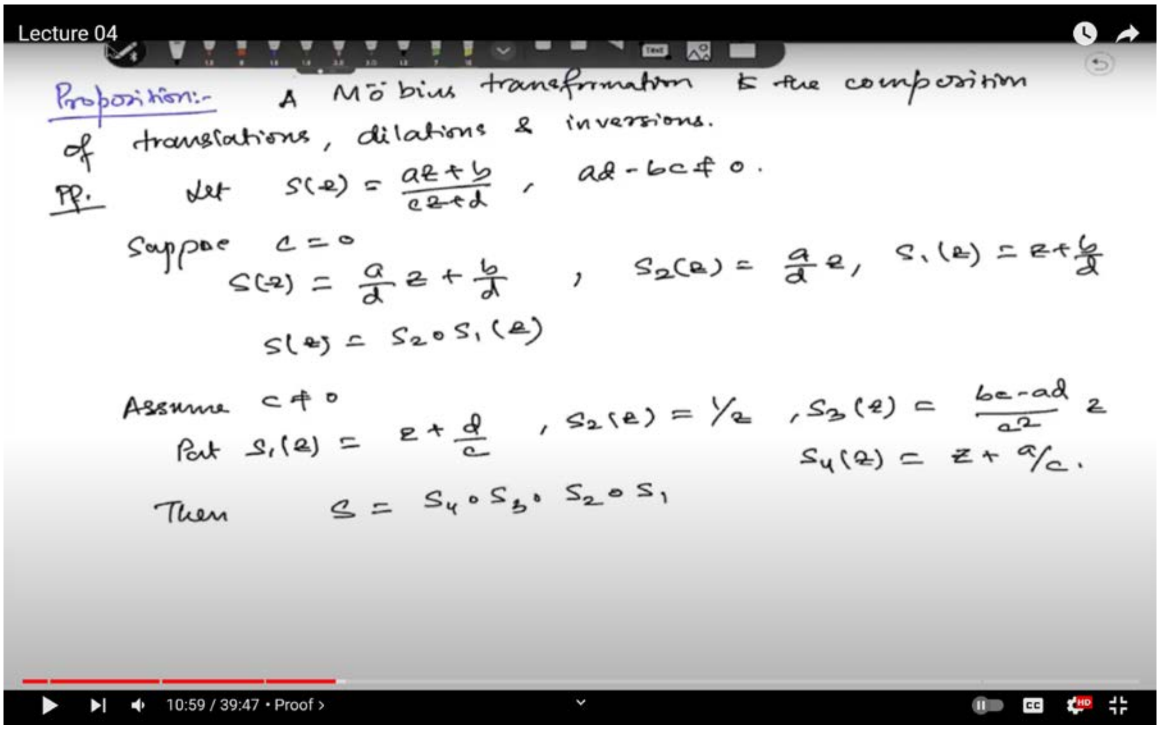
Let me prove the following proposition: A Möbius transformation can be expressed as a composition of translations, dilations, and inversions. Now, let's go through the proof of this.

Consider a Möbius transformation S(z), defined as:

$$S(z) = \frac{Az + B}{Cz + D}$$

where $AD - BC \neq 0$.

(Refer Slide Time: 10:59)



Case 1: When C = 0

If C = 0, the transformation simplifies to:

$$S(z) = \frac{A}{D}z + \frac{B}{D}$$

This is simply a combination of a dilation and a translation. Specifically, we can rewrite this as:

$$S(z) = S_2 \circ S_1(z)$$

where $S_1(z) = z + \frac{B}{D}$ represents a translation, and $S_2(z) = \frac{A}{D}$ z represents a dilation.

Case 2: When $C \neq 0$

Now, suppose $C \neq 0$. In this case, we perform the following steps:

1. First, apply the translation:

$$S_1(z) = z + \frac{D}{C}$$

2. Then, apply the inversion:

$$S_2(z) = \frac{1}{z}$$

3. Next, apply the dilation:

$$S_3(z) = \frac{BC - AD}{C^2} z$$

4. Finally, apply another translation:

$$S_4(z) = z + \frac{A}{C}$$

Thus, the original transformation S(z) can be written as:

$$S(z) = S_4 \circ S_3 \circ S_2 \circ S_1(z)$$

In conclusion, we see that any Möbius transformation can indeed be expressed as a composition of translations, dilations, and inversions.

Now, let's move on to the fixed points of a Möbius transformation. Suppose S is a Möbius transformation given by:

$$S(z) = \frac{Az + B}{Cz + D}$$

To find the fixed points of S, we need to solve the equation S(z) = z, which leads to:

$$\frac{Az + B}{Cz + D} = z$$

Multiplying both sides by Cz + D, we get:

$$Az + B = z(Cz + D)$$

Simplifying further, this results in the quadratic equation:

$$Cz^2 + (D - A)z - B = 0$$

This quadratic equation can have at most two solutions, meaning that S(z) can have at most two fixed points.

Next, let's consider the action of a Möbius transformation on three distinct points. Let A, B, C be three distinct points in C_∞, and let S be a Möbius transformation such that:

$$S(A) = \alpha, \quad S(B) = \beta, \quad S(C) = \gamma$$

Now, suppose T is another Möbius transformation that also maps these three points to the same values:

$$T(A) = \alpha, \quad T(B) = \beta, \quad T(C) = \gamma$$

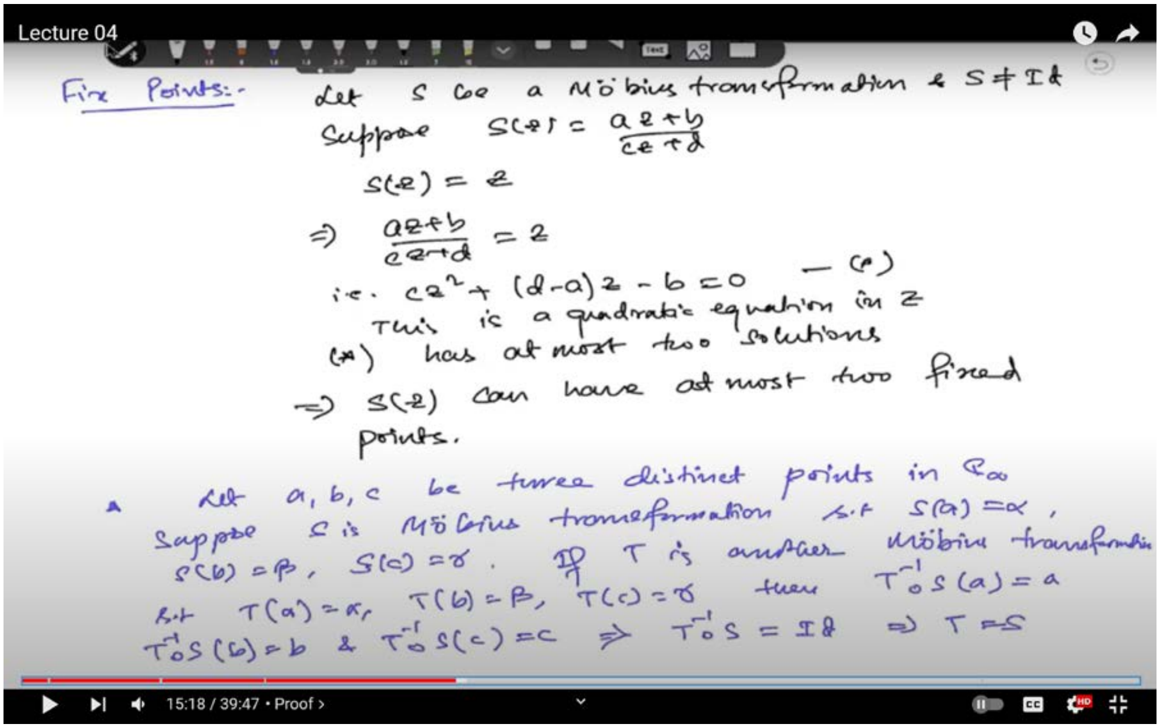
This implies that the composition $T^{-1} \circ S$ fixes the points A, B, and C. Since a non-identity Möbius transformation can have at most two fixed points, $T^{-1} \circ S$ must be the identity map. Therefore, we conclude that:

$$T = S$$

Thus, a Möbius transformation is uniquely determined by its action on any three distinct points in C_∞.

Let us now build upon the previous discussion. Consider a Möbius transformation S, and we assume that S is not the identity transformation. Now, let's take three distinct points Z_2, Z_3, Z_4 from C_∞ (the extended complex plane).

(Refer Slide Time: 15:18)



We define the transformation S as a map from C_∞ to itself. Specifically, for a point $Z \in C_\infty$, the Möbius transformation S(Z) is given by the expression:

$$S(Z) = \frac{(Z - Z_3)(Z_2 - Z_4)}{(Z - Z_4)(Z_2 - Z_3)}$$

Here, Z_2, Z_3, Z_4 are distinct points in C_∞, and this expression captures the transformation's structure.

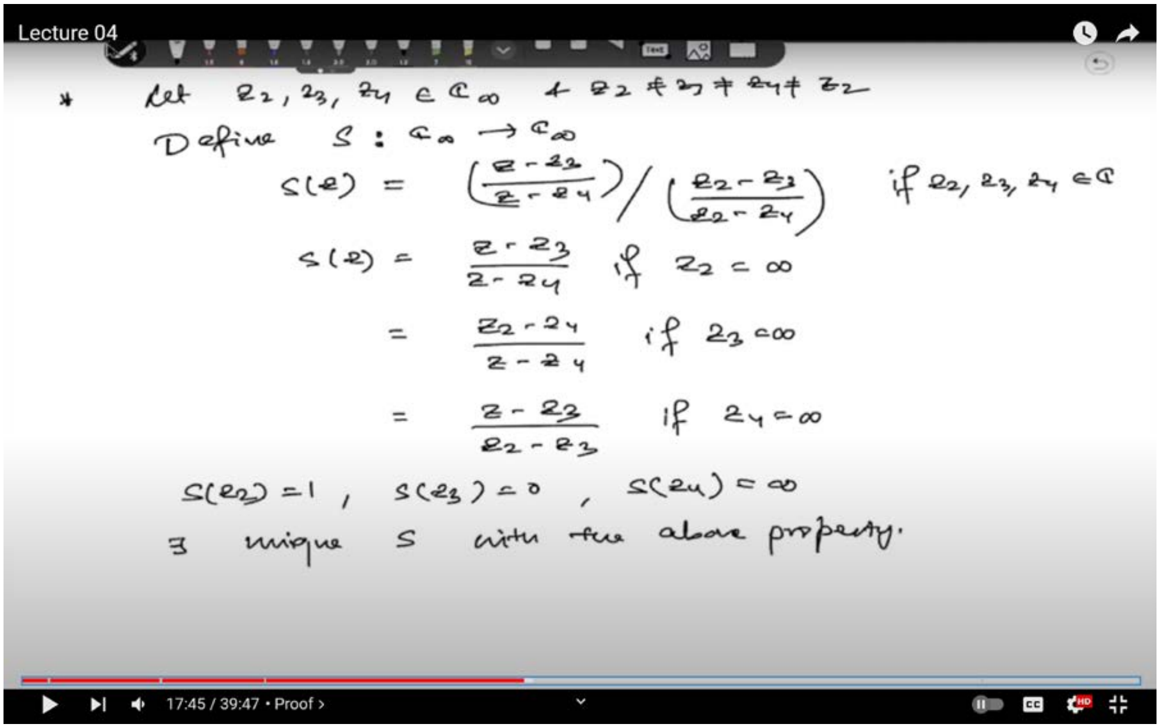
If $Z_2, Z_3, Z_4 \in C$ (that is, they are finite complex numbers), the transformation simplifies to:

$$S(Z) = \frac{Z - Z_3}{Z - Z_4}$$

In the case where $Z_2 = \infty$, the Möbius transformation is defined as:

$$S(Z) = \frac{Z_2 - Z_4}{Z - Z_4}$$

(Refer Slide Time: 17:45)



Now, if $Z_3 = \infty$, the transformation simplifies to:

$$S(Z) = \frac{Z - Z_4}{Z_2 - Z_4}$$

Similarly, if $Z_4 = \infty$, the transformation is defined as:

$$S(Z) = \frac{Z - Z_3}{Z_2 - Z_3}$$

This is the basis for the Möbius transformation S(Z).

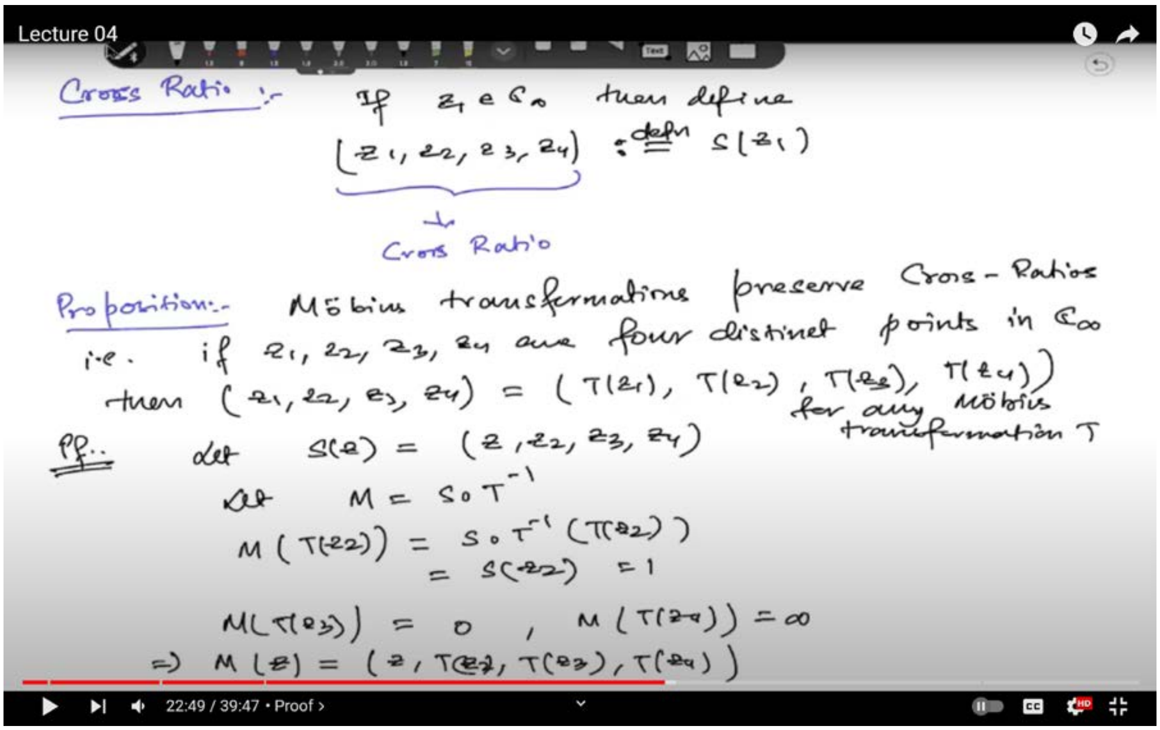
Next, we demonstrate the specific mappings under this transformation:

- $S(Z_2) = 1$
- $S(Z_3) = 0$
- $S(Z_4) = \infty$

From the earlier observations, there exists a unique Möbius transformation with this property.

Thus, for any three distinct points in the complex plane, there is a Möbius transformation that maps those three points to 1, 0, and ∞, respectively. Moreover, this Möbius transformation is unique, determined solely by the given points.

(Refer Slide Time: 22:49)



Alright, now let's move on to the concept of the cross ratio. Suppose $Z_1 \in C_\infty$ (the extended complex plane). The cross ratio is defined as follows:

$$[Z_1, Z_2, Z_3, Z_4] = S(Z_1)$$

where S is a Möbius transformation, as defined earlier. So, $S(Z_1)$ is, in fact, the cross ratio and is denoted by $[Z_1, Z_2, Z_3, Z_4]$.

This is known as the cross ratio of the points Z_1, Z_2, Z_3, Z_4. It can be written more explicitly in terms of Z_1 and the other points, but the key takeaway is that this is the definition of the cross ratio. Now, we state the next important proposition:

Proposition: Möbius transformations preserve cross ratios.

In other words, if you have four distinct points Z_1, Z_2, Z_3, Z_4 in C_∞, then for any Möbius transformation T, the cross ratio is preserved. Mathematically, this means:

$$[Z_1, Z_2, Z_3, Z_4] = [T(Z_1), T(Z_2), T(Z_3), T(Z_4)]$$

Proof:

Let's set the cross ratio [Z_1, Z_2, Z_3, Z_4] equal to some value λ. The goal is to show that applying any Möbius transformation T preserves this cross ratio.

We begin by considering a specific Möbius transformation S, which represents the cross ratio [Z, Z_2, Z_3, Z_4]. Now, let's define a new Möbius transformation:

$$M = S \circ T^{-1}$$

Since M is a Möbius transformation, we now apply it to T(Z_2):

$$M\big(T(Z_2)\big) = 1 \quad \text{(since S(Z}_2\text{) = 1)}$$

Similarly, we calculate:

$$M\big(T(Z_3)\big) = 0 \quad \text{(since S(Z}_3\text{) = 0)}$$

$$M\big(T(Z_4)\big) = \infty \quad \text{(since S(Z}_4\text{) = }\infty\text{)}$$

Thus, M maps T(Z_2) to 1, T(Z_3) to 0, and T(Z_4) to ∞.

From the uniqueness property of Möbius transformations, we know that if a Möbius transformation maps three distinct points to 1, 0, and ∞, then it must be of the form that preserves the cross ratio. Therefore, we have:

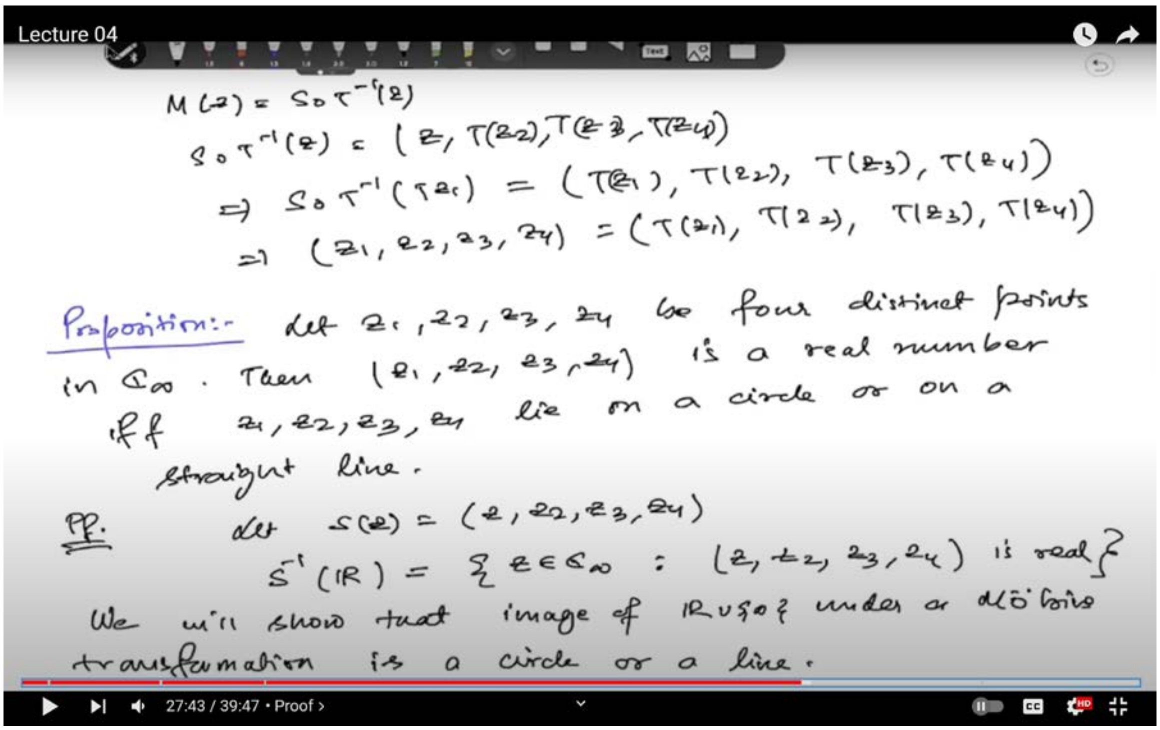
$$M(Z) = [Z, T(Z_2), T(Z_3), T(Z_4)]$$

Since $M = S \circ T^{-1}$, we conclude that:

$$S\left(T^{-1}(Z)\right) = [Z, T(Z_2), T(Z_3), T(Z_4)]$$

Hence, the cross ratio is preserved under any Möbius transformation T. This completes the proof that Möbius transformations preserve cross ratios.

(Refer Slide Time: 27:43)



So, this implies that if we substitute T(Z_1) in place of Z, the expression simplifies. On the right-hand side, we have [T(Z_1), T(Z_2), T(Z_3), T(Z_4)], which is the cross ratio of the transformed points. On the left-hand side, this corresponds to S(Z_1), which is simply the original cross ratio [Z_1, Z_2, Z_3, Z_4]. Therefore, we conclude that:

$$[Z_1, Z_2, Z_3, Z_4] = [T(Z_1), T(Z_2), T(Z_3), T(Z_4)]$$

This completes the proof of the proposition that Möbius transformations preserve the cross ratio.

Now, we move on to a particularly crucial result. We will prove that if the cross ratio is a real number, then the four points corresponding to the cross ratio lie either on a circle or on a straight line. The converse of this statement is also true. Let's formalize this in a proposition:

Proposition: Let Z_1, Z_2, Z_3, Z_4 be four distinct points in C_∞. The cross ratio [Z_1, Z_2, Z_3, Z_4] is a real number if and only if the four points Z_1, Z_2, Z_3, Z_4 lie on a circle or a straight line.

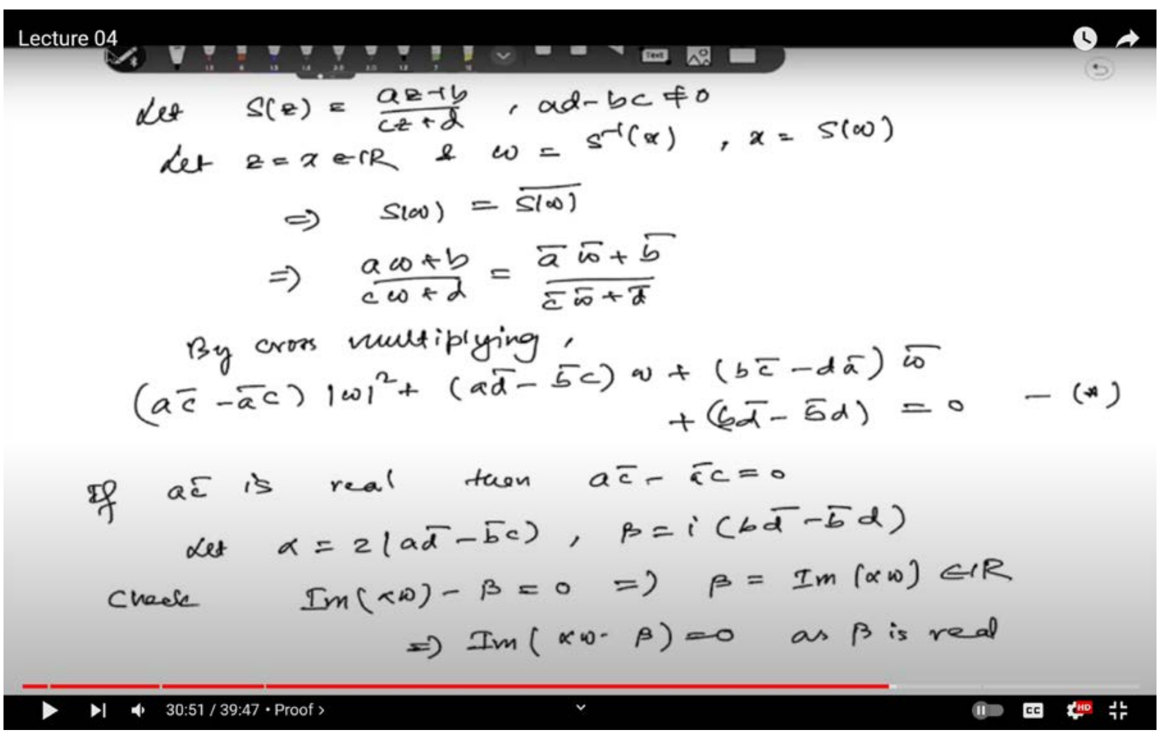
Proof:

Let's begin by considering a Möbius transformation S such that:

$$S(Z) = \frac{Z - Z_3}{Z - Z_4}$$

By the definition of the inverse image, the set $S^{-1}(R)$ (where R denotes the real line) is given by the set of all complex numbers $Z \in C_\infty$ such that the cross ratio [Z, Z_2, Z_3, Z_4] is a real number.

(Refer Slide Time: 30:51)



Now, since S^{-1} is also a Möbius transformation, if we can show that the inverse image $S^{-1}(R)$ is either a circle or a line, we will have proven the desired result.

We aim to demonstrate that the image of $R \cup \infty$ under a Möbius transformation is always either a circle or a line. Once this is shown, it follows that the set of points for which the cross ratio is real must lie on a circle or a straight line. This completes the proof.

Let us now consider the Möbius transformation $S(z) = \frac{az+b}{cz+d}$, where $ad - bc \neq 0$. This is our Möbius transformation. Now, let's assume $z = x \in R$ and let w be such that $w = S^{-1}(x)$. Therefore, x = S(w), and since x is real, this means that S(w) is also a real number.

Thus, we have:

$$S(w) = \frac{aw + b}{cw + d} = \overline{\left(\frac{aw + b}{cw + d}\right)}$$

This implies that:

$$\frac{aw + b}{cw + d} = \frac{\overline{a}\overline{w} + \overline{b}}{\overline{c}\overline{w} + \overline{d}}$$

By cross-multiplying both sides, we obtain a more detailed expression:

$$(a\overline{c} - \overline{a}c)|w|^2 + \left(a\overline{d} - \overline{b}c\right)w + (b\overline{c} - d\overline{a})\overline{w} + \left(b\overline{d} - \overline{b}d\right) = 0$$

Now, let's simplify this equation. First, if $a\overline{c}$ is a real number, then:

$$a\overline{c} = \overline{a\overline{c}}$$

This implies that $a\overline{c} - \overline{a}c = 0$, and therefore, the coefficient of $|w|^2$ vanishes. We can now rewrite the remaining terms using new variables for clarity. Define:

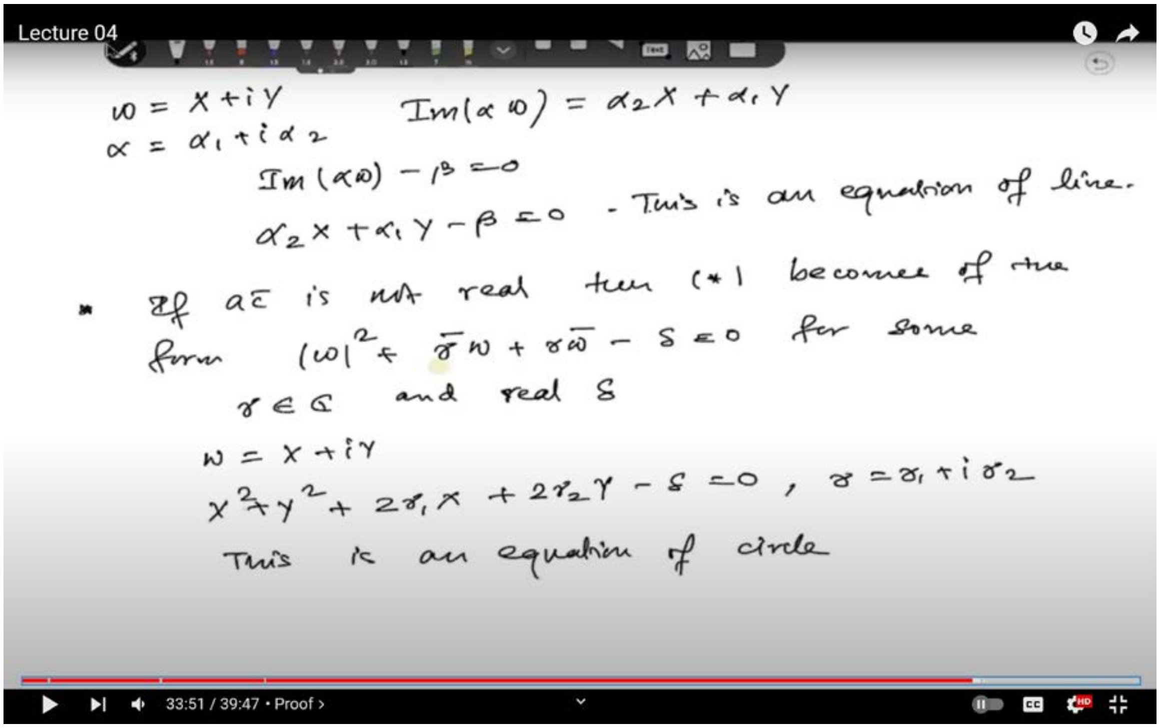
$$\alpha = 2\left(a\overline{d} - \overline{b}c\right) \quad \text{and} \quad \beta = i\left(b\overline{d} - \overline{b}d\right)$$

It can be shown that the imaginary part of $\alpha w - \beta$ equals zero, which implies:

$$\text{Im}(\alpha w - \beta) = 0$$

This result confirms that β, which is the imaginary part of α w, is indeed a real number. Therefore, the imaginary part of α w - β vanishes, further solidifying the real nature of the transformation in this scenario.

(Refer Slide Time: 33:51)



So, we have β = 0 and β is a real number. Now, let us express w as W = X + iY and α as α = α_1 + iα_2. Then, the imaginary part of α w becomes:

$$\text{Im}(\alpha w) = \alpha_2 X + \alpha_1 Y$$

Since we know that the imaginary part of α w - β is equal to zero, and β is a real number, we arrive at the equation:

$$\alpha_2 X + \alpha_1 Y - \beta = 0$$

This is the equation of a straight line. Therefore, when $a\bar{c}$ is real, we are done; the points lie on a straight line.

Now, let's consider the case where $a\bar{c}$ is not real. In this scenario, the equation (denoted by ⋆) becomes:

$$|w|^2 + \bar{\gamma}w + \gamma\bar{w} - \delta = 0$$

where $\gamma \in C$ and $\delta \in R$. The fact that δ is real is straightforward since the middle term must be real. If we let this middle term be p, we have $p = \bar{p}$, which guarantees its real nature.

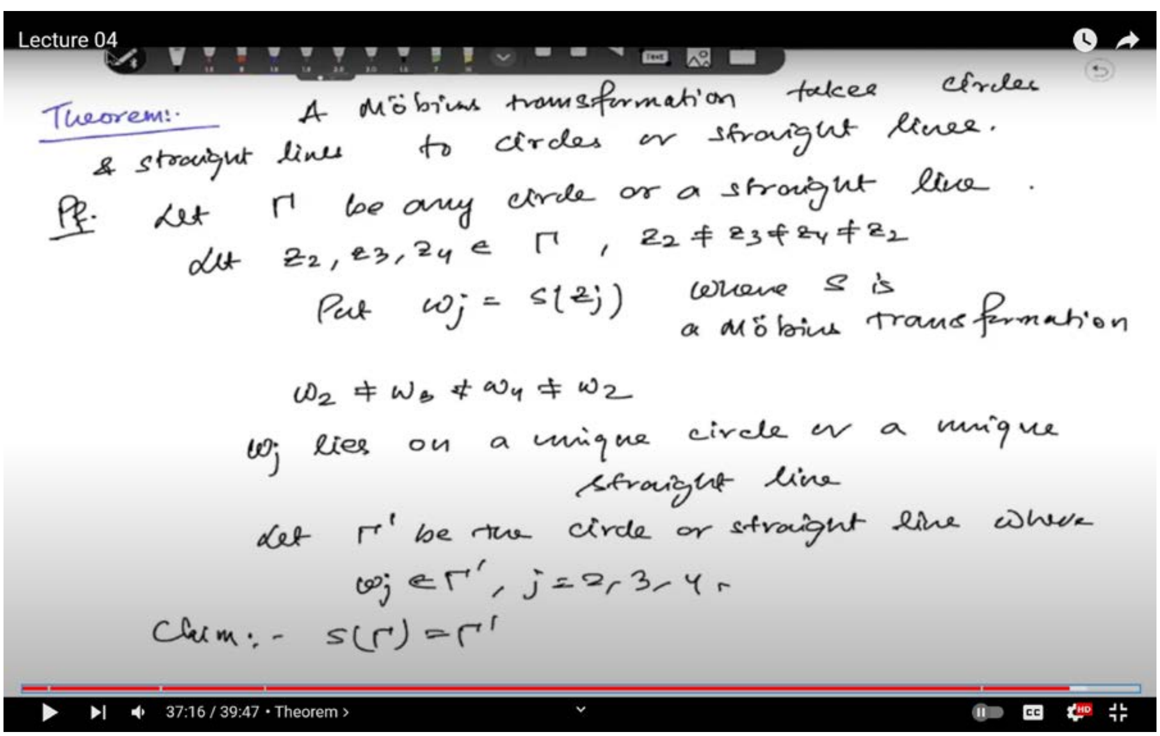
Now, express w = x + iy. Substituting into the equation, we get:

$$x^2 + y^2 + 2\gamma_1 x + 2\gamma_2 y - \delta = 0$$

where $\gamma = \gamma_1 + i\gamma_2$. This is clearly the equation of a circle.

Thus, we have proven the proposition: if we take any four distinct points and if the cross ratio [z_1, z_2, z_3, z_4] is a real number, then these four points lie on either a circle or a straight line. Furthermore, the converse is also true. With this, the proposition is proven. Now, let's move on to the next important proposition.

(Refer Slide Time: 37:16)



So, it is stated that a Möbius transformation maps circles to either a straight line or another circle. Similarly, if you take a straight line, its image under a Möbius transformation will also be either a circle or a straight line. Let's formalize this concept as a theorem:

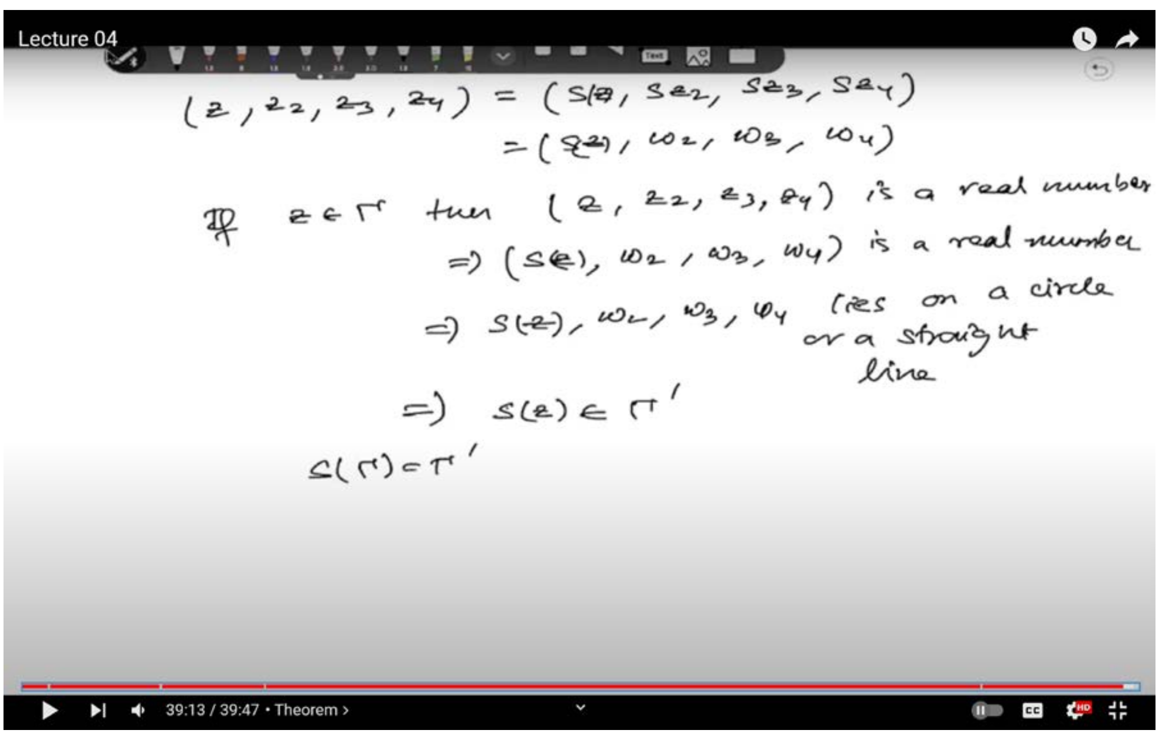
Theorem: A Möbius transformation takes circles and straight lines to circles or straight lines.

Now, let's prove this theorem. Let γ be any circle or straight line. We will consider three distinct points z_2, z_3, z_4 that lie on γ, ensuring they are distinct.

Next, we define w_j to be $S(z_j)$, where S is our Möbius transformation. Since S is a bijection, the points w_j are also distinct. Therefore, we now have three distinct points w_2, w_3, w_4.

Now, we must consider the configuration of these three points. If they are collinear, then there exists a straight line that passes through all three distinct points. Conversely, if they are not collinear, there exists a unique circle that passes through these three points w_2, w_3, w_4.

(Refer Slide Time: 39:13)



Let us denote the circle or straight line formed by these three points as γ'. Now, the crucial claim we need to establish is that the image of γ under the Möbius transformation S is equal to γ':

$$S(\gamma) = \gamma'$$

This assertion is what we aim to prove.

We know that a Möbius transformation preserves the cross ratio. Specifically, if we take the cross ratio (z, z_2, z_3, z_4), it is equal to the cross ratio $(S(z), S(z_2), S(z_3), S(z_4))$. This can also be expressed as $S(z, w_2, w_3, w_4)$.

Now, if z belongs to γ, then the left-hand side of the equation is a real number. According to our earlier observations, this implies that $S(z, w_2, w_3, w_4)$ is also a real number. By the previous proposition, we know that if this expression is a real number, then it must lie on either a circle or a straight line.

Given that there is a unique circle or straight line passing through the points w_2, w_3, and w_4, we can conclude that S(z) must also belong to γ'. Thus, we have successfully proved that

$$S(\gamma) = \gamma'.$$

With that established, we are done with this part of the proof. In the next class, we will prove that Möbius transformations are conformal, meaning they preserve angles. After that, we will move on to discuss the geodesics of the upper half-plane. I'll stop here for now.