An Introduction to Hyperbolic Geometry

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Lecture – 39

Finite Conjugacy Classes and Quasi-Convexity in Hyperbolic Groups

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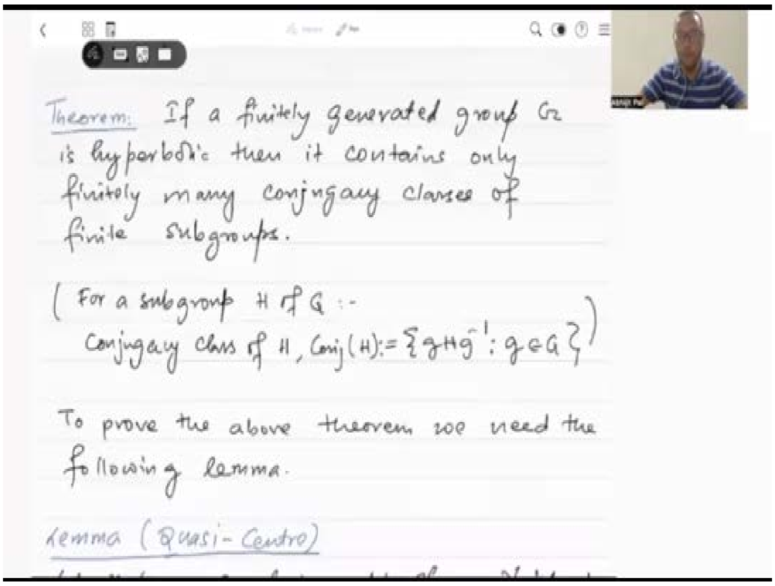

Hello! In this lecture, we will demonstrate that a hyperbolic group contains only finitely many conjugacy classes of finite subgroups. Following this, we will introduce the concept of quasi-convex subgroups within a hyperbolic group.

Let's begin with this theorem: Let G be a hyperbolic group. According to the definition of a hyperbolic group, G must be finitely generated, and the triangles in its Cayley graph must exhibit a slim configuration.

Now, if we consider a finitely generated group, it follows that this group G contains only finitely many conjugacy classes of finite subgroups. But first, let's clarify what we mean by the conjugacy class of a subgroup. The conjugacy class of a subgroup H is simply the set of all conjugates of the subgroup H.

To prove this theorem, we first need the following lemma. Let X be a geodesic metric space that is δ-hyperbolic, and let Y be a non-empty bounded closed set within X. Our objective is to identify a point, which we will refer to as the quasi-center of this bounded set.

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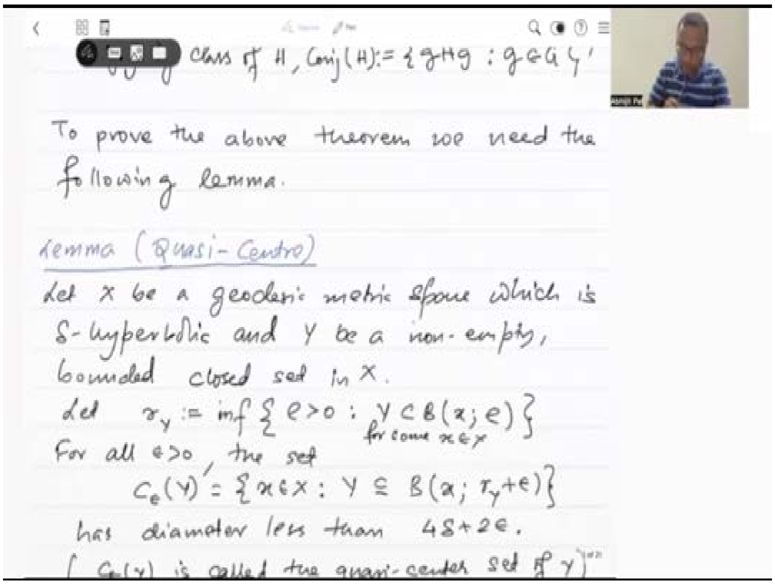

How do we go about this? We define r_Y as the infimum of all positive numbers ρ such that Y is contained within some ball centered at a point x in X with radius ρ.

Next, we introduce the set $C_\epsilon(Y)$, where ϵ is a positive number. The set $C_\epsilon(Y)$ consists of all elements x in X such that Y is contained within the ball centered at x with radius $r_Y + \epsilon$.

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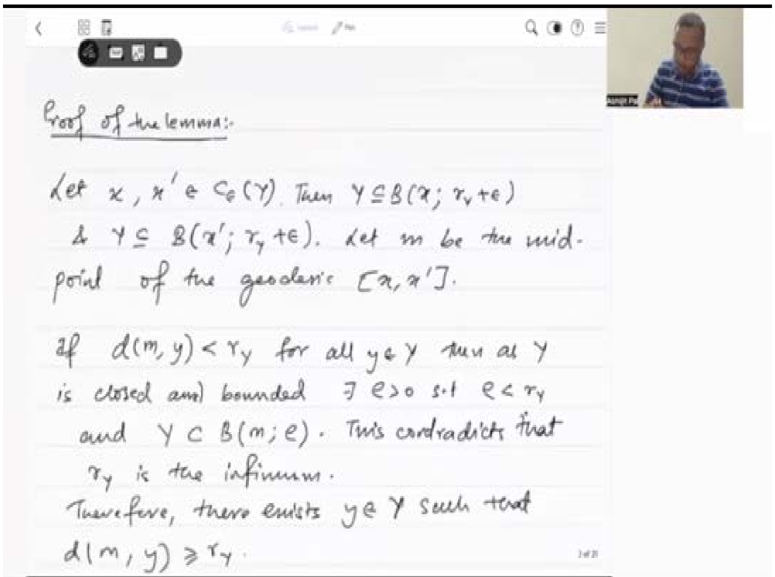

We aim to demonstrate that the set $C_\epsilon(Y)$ has a diameter of less than $4\delta + 2\epsilon$. Furthermore, we will show that this collection of points, denoted as $C_\epsilon(Y)$, is a bounded set. Any point within this set will be referred to as a quasi-center. Now, let us proceed with the proof of this lemma.

Let us consider two points, x and x', within the set $C_\epsilon(Y)$. By definition, this means that Y is contained within the ball of radius $r_Y + \epsilon$ centered around x, as well as within the ball of radius $r_Y + \epsilon$ centered around x'.

Now, to establish a connection between these two points, we take m to be the midpoint of the geodesic that joins x and x'. This choice of m allows us to explore the geometric properties of the space more effectively.

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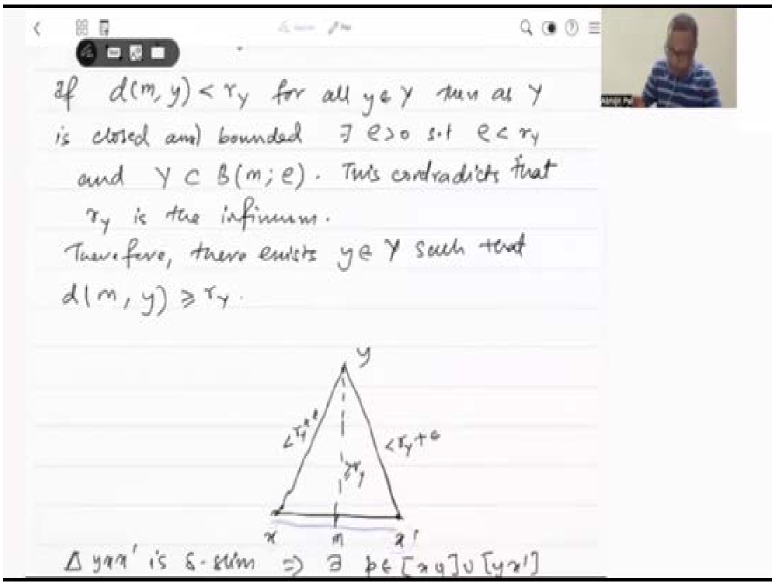

Here is a visual representation of our situation: we have points x and x' located in the set $C_\epsilon(Y)$, with m serving as the midpoint of the geodesic connecting x and x'. Now, consider the distance from this midpoint m to every point in Y. If this distance is strictly less than r_Y, then, given that Y is closed and bounded, there must exist some positive number ρ such that ρ is also strictly less than r_Y.

Consequently, we would find that Y is contained within the ball of radius ρ centered at m. However, this creates a contradiction, as it implies that r_Y cannot be the infimum. Therefore, our initial assumption must be incorrect. This leads us to conclude that there exists at least one element y in the bounded set Y such that the distance between m and y is greater than or equal to r_Y.

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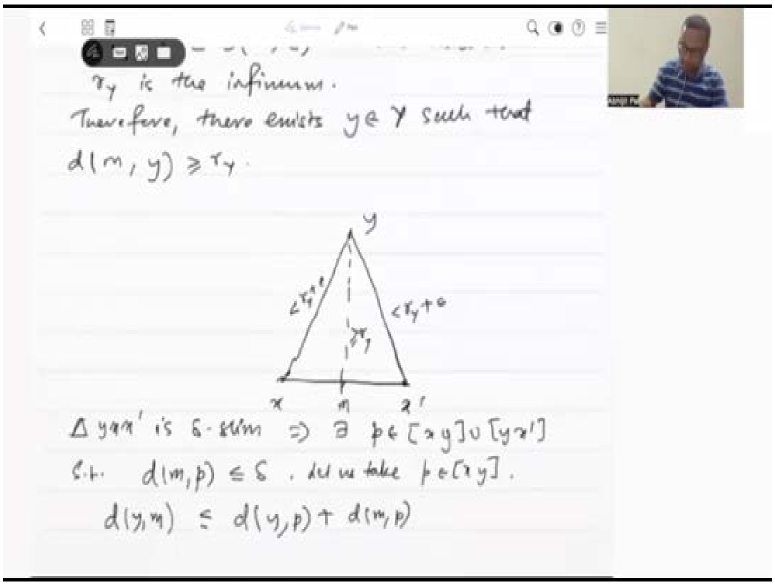

Let's visualize the situation at hand. We have established that the distance between the midpoint m and the point y is greater than or equal to r_Y. Now, let's analyze the triangle formed by the points y, x, and x'. This triangle is considered δ-slim, as x serves as a hyperbolic vertex base.

For the point m, there exists a point p that lies either on the geodesic connecting x and y or on the geodesic connecting x' and y, such that the distance between m and p is less than or equal to δ.

Let's first consider the case where p belongs to the geodesic joining x and y. In this scenario, we can state that the distance from y to m is less than or equal to the distance from y to p plus the distance from m to p.

Now, let's analyze the inequality we have established: r_Y is less than or equal to the distance between y and m, which is also less than or equal to the distance from y to p plus the distance from p to m, and importantly, this distance is again less than or equal to δ.

This leads us to an important conclusion: the distance between y and p is equal to the distance from x to y minus the distance from x to p. This is evident from our diagram since p lies on the geodesic.

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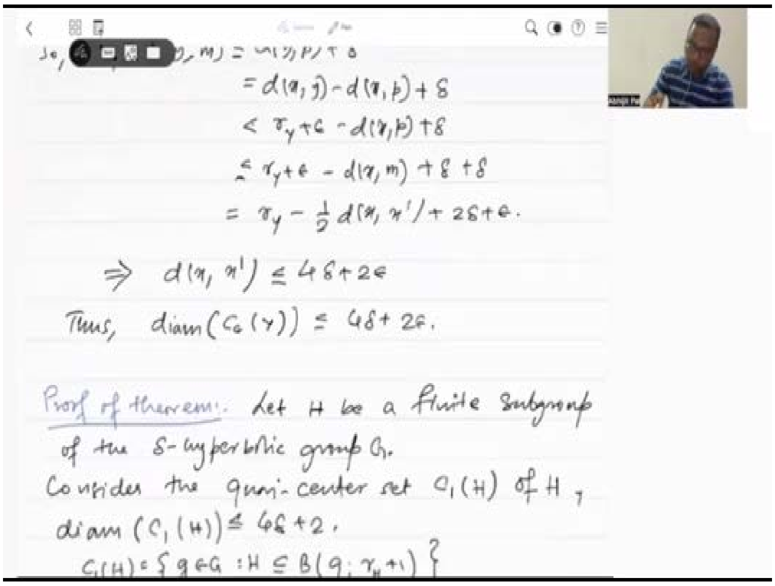

Furthermore, we know that the distance between x and y is strictly less than $r_Y + \epsilon$, which follows from the fact that y is an element of the set Y. Thus, we establish the inequality: the distance from x to y is strictly less than $r_Y + \epsilon$.

Now, applying the triangle inequality, we can express this relationship as $-\text{distance}(x, p) \le -\text{distance}(x, m) + \delta$. Given that the distance between x and m is half of the distance between x and x' (since m is the midpoint of the geodesic connecting x and x'), we can reformulate our expression.

What we ultimately find is that r_Y is strictly less than $r_Y - \frac{1}{2}\text{distance}(x, x') + 2\delta + \epsilon$. Consequently, we can derive that the distance between x and x' is less than or equal to $4\delta + 2\epsilon$.

Thus, we conclude that the diameter of the set $C_\epsilon(Y)$ is indeed less than or equal to $4\delta + 2\epsilon$. Now, let's return to the proof of the theorem.

Let us consider a finite subgroup H within the hyperbolic group G. Here, I will denote the hyperbolic constant of this group as δ. Our goal is to demonstrate that H possesses only finitely many conjugacy classes.

To begin, we will examine the quasi-center set $C_1(H)$, where ϵ is set to 1. According to our earlier lemma, the diameter of $C_1(H)$ is less than or equal to $4\delta + 2$.

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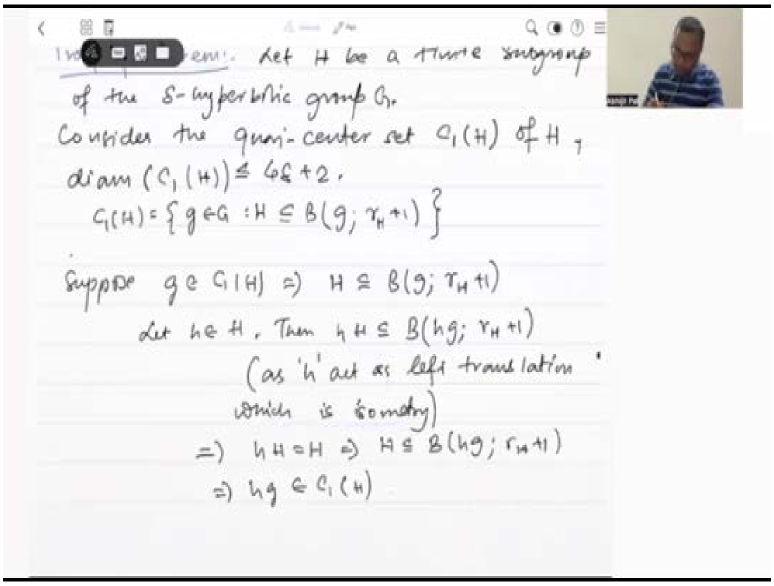

Now, let's recall the definition of $C_1(H)$: it consists of all group elements g such that H is contained within the ball of radius $r_H + 1$ centered at g.

When we take an element g from $C_1(H)$, it implies that H is indeed a subset of this ball. Now, consider any element h from the subgroup H. It's crucial to note that the left coset hH, which is essentially H itself, is contained in the ball of radius $r_H + 1$ around the point hg.

This holds true because left translation by the element h is an isometric operation. Therefore, we conclude that the subgroup H remains contained within this ball of radius $r_H + 1$ centered at hg.

Consequently, this implies that hg also belongs to $C_1(H)$. Thus, we have established that starting from an element g in $C_1(H)$, we have shown that hg is contained in $C_1(H)$ for every element h in the subgroup H.

Thus, we have established that the subgroup H leaves $C_1(H)$ invariant. Now, let's select a vertex x within $C_1(H)$. It can be verified that if we take the conjugate $x^{-1}Hx$, this operation leaves $x^{-1}C_1(H)$ invariant. I encourage you to check this assertion.

Next, notice that the identity element 1 belongs to $x^{-1}C_1(H)$ because we have chosen x to be in $C_1(H)$. Consequently, the diameter of $x^{-1}C_1(H)$ is less than or equal to $4\delta + 2$. This holds true because the operation of x^{-1} acts by left translation, which is an isometric operation.

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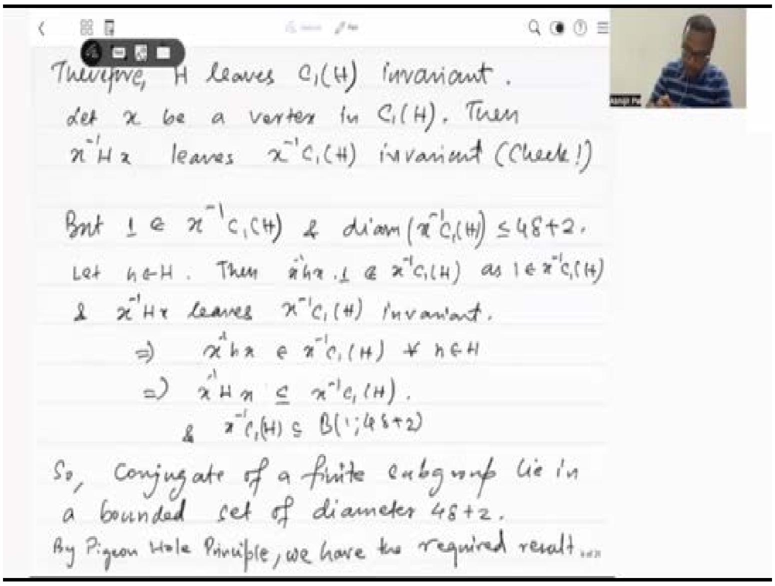

Now, let's consider an element h from the subgroup H. It is important to observe that the conjugate $x^{-1}Hx$ belongs to the subgroup $x^{-1}H\,x$, which in turn is contained in $x^{-1}C_1(H)$. This relationship is valid because the conjugate $x^{-1}Hx$ leaves $x^{-1}C_1(H)$ invariant.

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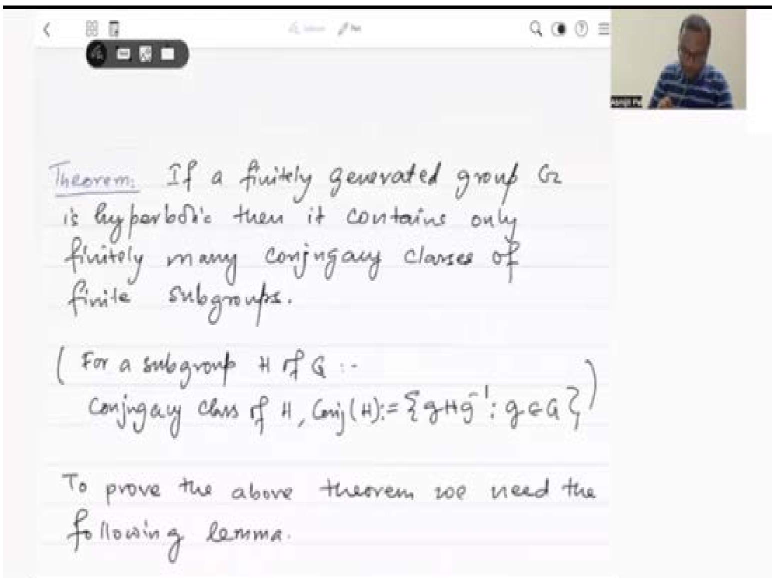

Thus, we have shown that $x^{-1}Hx$ is indeed a subset of $x^{-1}C_1(H)$. Moreover, we have established that $x^{-1}C_1(H)$ is contained within the ball of radius 4δ + 2 centered at the identity element 1.

This leads us to the conclusion that the conjugates of a finite subgroup lie within a bounded set with a diameter of 4δ + 2. As a result, applying the Pigeonhole Principle, we arrive at the desired outcome. What is this result, you may wonder?

Therefore, we have established that if we consider a finite subgroup H, it contains only finitely many conjugacy classes of that subgroup. This conclusion is an essential part of our proof.

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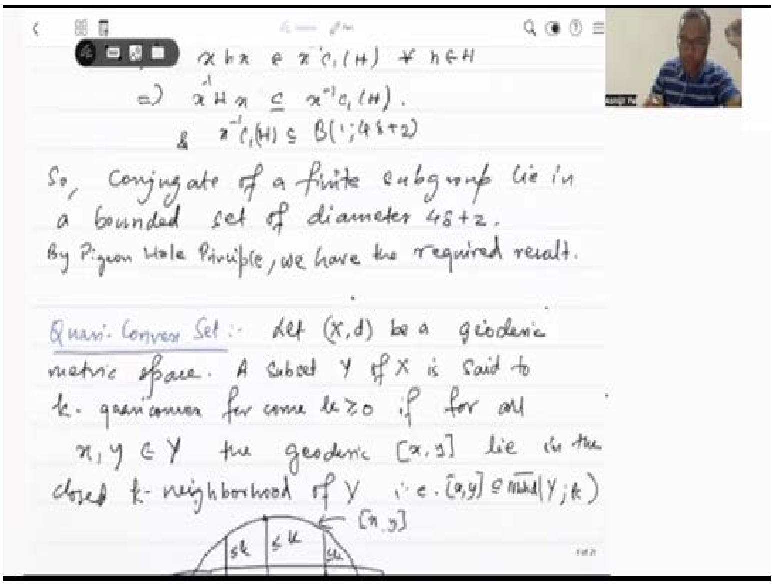

Since we are working with a finite set, specifically within the context of a finitely generated group, it follows that any finite ball will also be a finite set. Consequently, by the Pigeonhole Principle, we cannot have infinitely many conjugacy classes. Now, let's define what we mean by a quasi-convex set.

Let X be a geodesic metric space. A subset Y of X is said to be k-quasi-convex for some non-negative number k if, for every pair of elements x and y in the subset Y, the geodesic connecting x and y lies within the closed k-neighborhood of Y. In simpler terms, if we visualize our metric space X and take any two points x and y within the subset Y, the geodesic that connects these two points must reside in a k-neighborhood of Y.

What does this entail? It means that for any point along the geodesic connecting x and y, there exists a corresponding point in Y such that the distance between these two points is less than or equal to k. Thus, every point on the geodesic can be approximated by points in Y within a distance of k.

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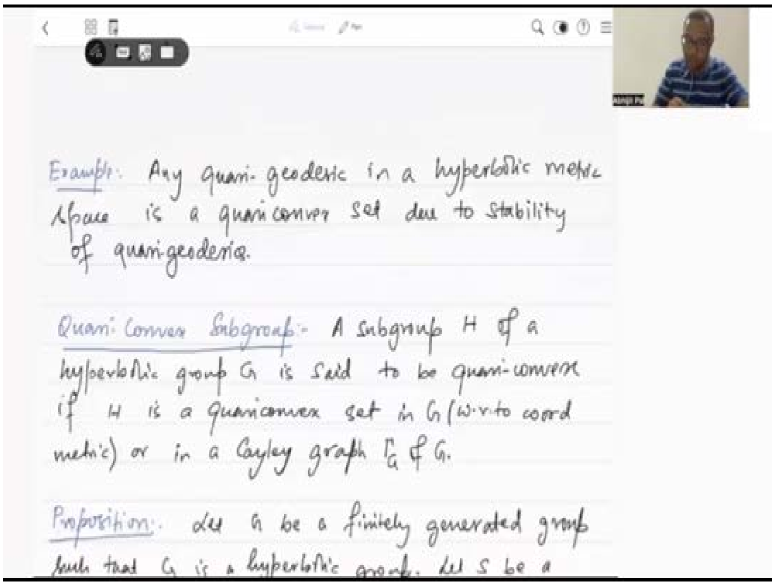

We previously discussed the definition of a quasi-geodesic. It can be shown that any quasi-geodesic within a hyperbolic metric space qualifies as a quasi-convex set. This assertion holds true due to the inherent stability of quasi-geodesics.

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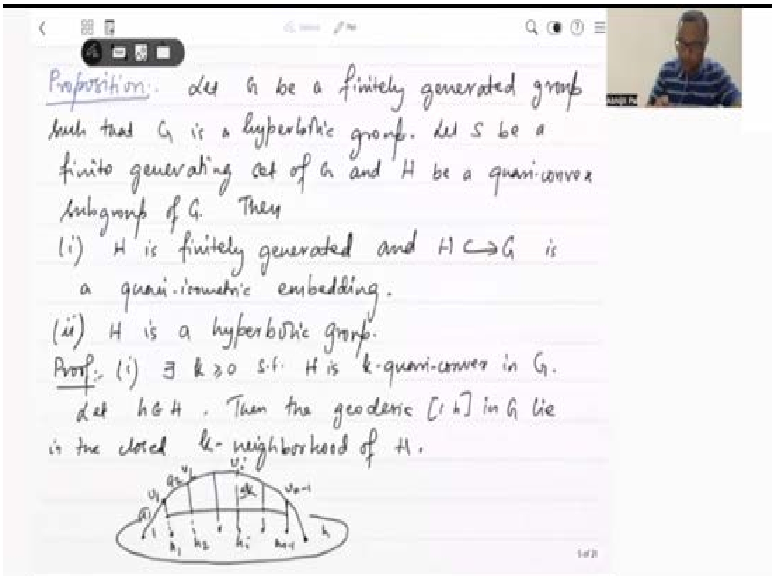

Now, let’s delve into the definition of a quasi-convex subgroup. A subgroup H of a hyperbolic group is classified as quasi-convex if H is a quasi-convex set in the Cayley graph of the group G. In other words, when we consider the Cayley graph and identify the subgroup H within it, that subgroup forms a subspace of the Cayley graph. If this subspace is indeed a quasi-convex set in the context of the Cayley graph, we refer to H as a quasi-convex subgroup.

Now, let's explore this proposition. Let G be a hyperbolic group, and let S be a finite generating set for G. Since G is a hyperbolic group, it is finitely generated, and we can indeed obtain a finite generating set for this group, which we have designated as S. Next, we consider H to be a quasi-convex subgroup of G. The claim we want to establish is that H is finitely generated and that the embedding of H into G is a quasi-isometric embedding. This will ultimately demonstrate that H is, in fact, a hyperbolic group.

To clarify, if we take any quasi-convex subgroup of a hyperbolic group, that subgroup will also be a hyperbolic group. Now, what we aim to show is that there exists some non-negative number k, but, just to clarify, we've already assumed that H is a quasi-convex subgroup. Therefore, there must exist a non-negative number k such that H is k-quasi-convex in G.

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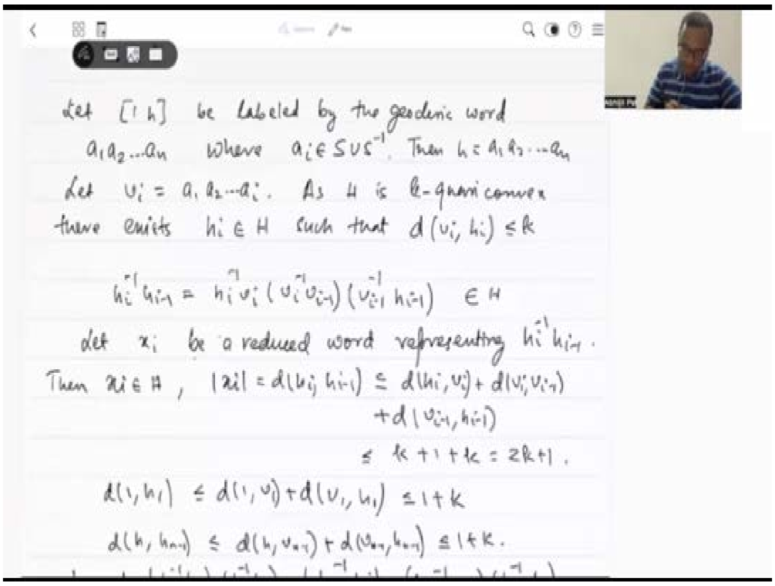

To proceed, let us take any element h in the subgroup H and consider the geodesic in G that connects the identity element 1 to h. Both 1 and h belong to the subgroup H. According to the definition of a quasi-convex subgroup, this geodesic will lie within the closed k-neighborhood of H.

So, if we visualize H, we can assert that this geodesic remains within the closed k-neighborhood of H. Therefore, for any point along this geodesic, there exists a corresponding point in the subgroup H such that the distance between these two points is less than or equal to k.

In this context, let's label the geodesic connecting the identity element 1 to the element h as a_1, a_2, …, a_n. Here, each a_i is either a generator or the inverse of a generator from our finite generating set S. Therefore, we can express H as the product a_1, a_2, …, a_n.

Next, we define v_i to be the product of the generators up to the i-th element, specifically $v_i = a_1 a_2 \dots a_i$. This gives us a clear representation of our structure, as illustrated in the accompanying diagram.

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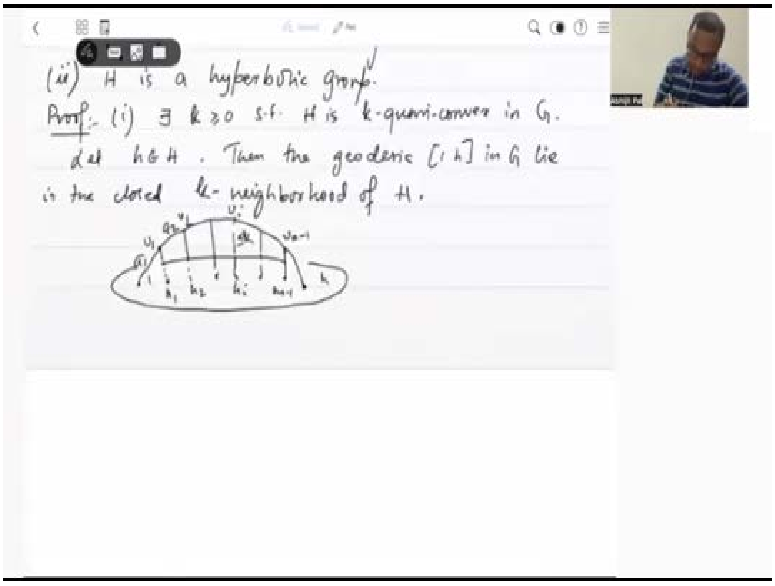

This illustration perfectly captures our discussion. Here, we have h represented as $a_1 a_2 \dots a_n$, where the first h is labeled as a_1 and the second as a_2. The vertex v_1 corresponds to a_1, while vertex v_2 represents the product $a_1 a_2$. Similarly, vertex v_i is denoted as $a_1 a_2 \dots a_i$.

For each vertex v_i along the geodesic associated with h, there exists a point h_i such that the distance between v_i and h_i is less than or equal to k. Given that h is quasi-convex, we find that h_i belongs to the subgroup H, maintaining this distance condition.

Now, if we consider the word $h_i^{-1} h_{i-1}$, it can be expressed as follows: $h_i^{-1} v_i v_i^{-1} h_{i-1}$. In other words, we can rewrite it as $v_i^{-1} h_{i-1}$. This reflects a path we can visualize based on our diagram.

Next, let us define x_i as a reduced word that represents $h_i^{-1} h_{i-1}$. This approach ties our geometric interpretation back to our algebraic representation effectively.

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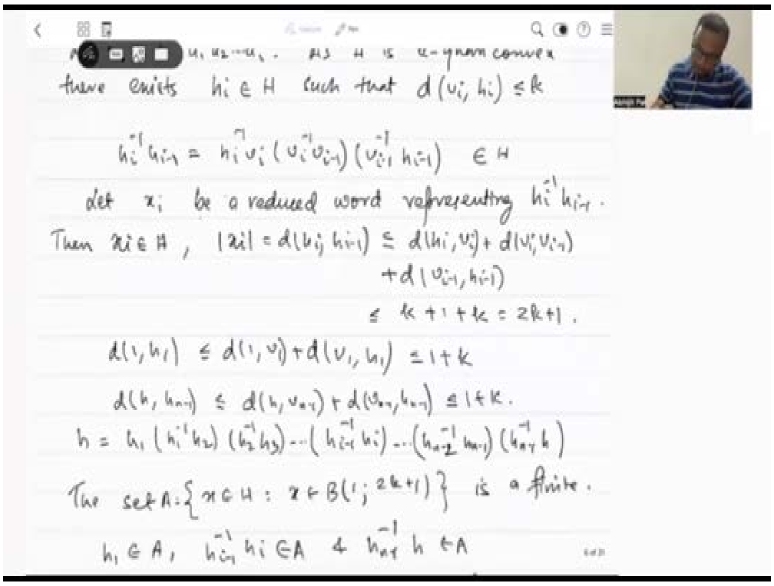

Since x_i is defined in terms of h_i and h_{i-1}, and both h_i and h_{i-1} belong to the subgroup H, we conclude that x_i also belongs to H. Now, let's analyze the length of x_i. Because x_i is a reduced word, its length is equal to the distance between h_i and h_{i-1}.

Next, we can apply the triangle inequality, which gives us the following relation: the length of x_i is less than or equal to the sum of three distances: the distance from h_i to v_i, the distance from v_i to v_{i-1}, and the distance from v_{i-1} to h_{i-1}. Mathematically, this can be expressed as:

$$\text{length}(x_i) \leq \text{distance}(h_i, v_i) + \text{distance}(v_i, v_{i-1}) + \text{distance}(v_{i-1}, h_{i-1}).$$

We know that the distance between h_i and v_i is less than or equal to k, and the distance between v_i and v_{i-1} is exactly 1. Additionally, the distance from v_{i-1} to h_{i-1} is also less than or equal to k. Thus, we can conclude:

$$\text{length}(x_i) \leq k + 1 + k = 2k + 1.$$

Moreover, considering the distance from the identity element 1 to h_1, we find that it is less than or equal to 1 + k. Similarly, at the other end of the geodesic, the distance between h_n and h_{n-1} is also less than or equal to 1 + k.

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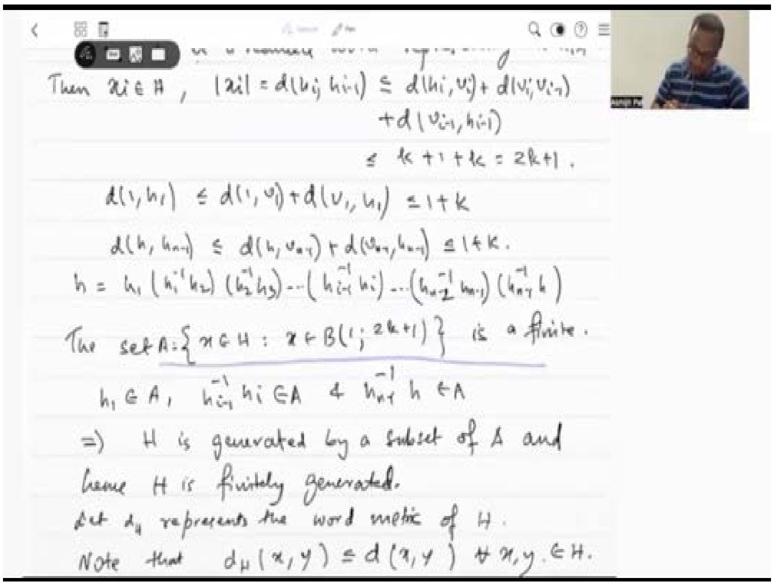

Now, let's observe that the element h can be expressed as a product of several components. Specifically, we can write h as follows:

$$h = h_1 \cdot h_1^{-1} \cdot h_2 \cdot h_2^{-1} \cdot h_3 \cdots h_{n-1}^{-1} \cdot h.$$

It's important to note that the lengths of each segment, $h_1, h_1^{-1}h_2, h_{i-1}^{-1}h_i$, are all less than or equal to 2k + 1.

Next, let's define the set A to be the collection of all elements x in the subgroup H such that x lies within the ball of radius 2k + 1 centered at the identity element 1. This set A is finite. Consequently, since all the elements h_1, h_{i-1}^{-1} h_i, and ultimately h_{n-1}^{-1} h belong to A, we can conclude that the subgroup H is generated by a subset of A. Because A is finite, we can assert that H itself is a finite degenerate group.

Now, let's delve into the distance metric associated with H. The distance d_H represents the word metric for the subgroup H. We have established a generating set for H, which allows us to define a separate word metric for this subgroup.

It is crucial to note that d_H, which signifies the distance within the group H, also serves as the word metric. For any elements x and y belonging to H, the distance in H satisfies the inequality:

$$d_H(x, y) \leq d_G(x, y),$$

where $d_G(x, y)$ is the distance in the larger group G. Since H is a subgroup of G and is finitely generated, it will have its own word metric d_H. Thus, we can confidently state that:

$$d_H(x, y) \leq \text{distance in the word metric of } G.$$

Please ensure that this relationship holds as we continue our examination.

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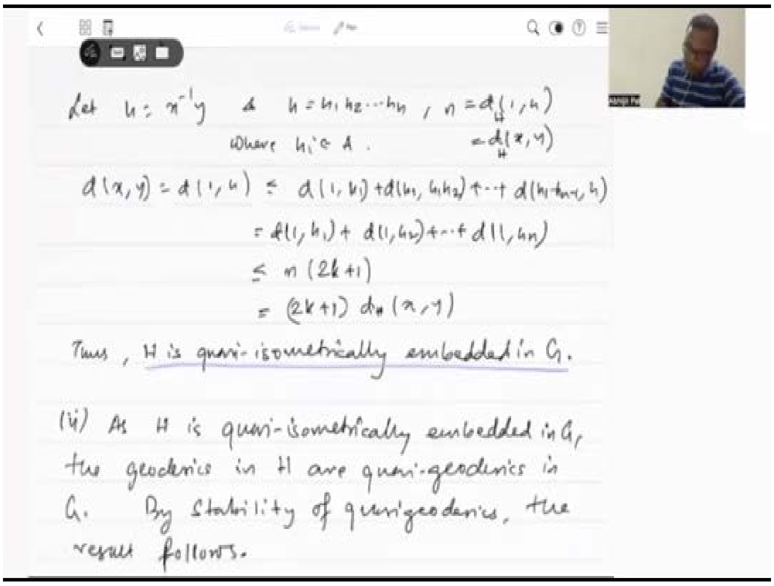
- 

Alright, let’s define h as $h = x^{-1}y$, and we can express h as a product of elements h_1, h_2, …, h_n, where each h_i belongs to the set A. Let’s denote n as the distance between the identity element 1 and h, measured within the subgroup H. Since we have defined n in this way, we can conclude that this word is indeed a reduced word.

Moreover, the distance between 1 and h will be equivalent to the distance between x and y, as we have $x^{-1}y = h$. It’s also important to note that the left action in this context is isometric.

Now, if we consider the distance between x and y in the group G, it can be expressed as the distance between 1 and h within G. By applying the triangle inequality, we find that this distance is less than or equal to the sum of distances:

$$\text{distance}(1, h_1) + \text{distance}(h_1, h_1 h_2) + \text{distance}(h_{n-1}, h).$$

Now, keep in mind that the left translation is isometric, meaning that the distance between h_1 and h_1h_2 is the same as the distance between 1 and h_2. Similarly, the distance between h_{n-1} and h (which is equal to $h_1h_2 \dots h_n$) will correspond to the distance between 1 and h_n.

Since each h_i belongs to the set A, we know that the distance between 1 and h_i is less than or equal to 2k + 1. Therefore, we can establish that:

$$\text{distance}(1, h) \leq n \times (2k + 1).$$

What does n represent? It signifies the distance in the subgroup H between x and y. Thus, we can summarize that the distance between x and y in G satisfies:

$$d_G(x, y) \leq (2k + 1) \times d_H(x, y).$$

From this, we conclude that H is quasi-isometrically embedded in G.

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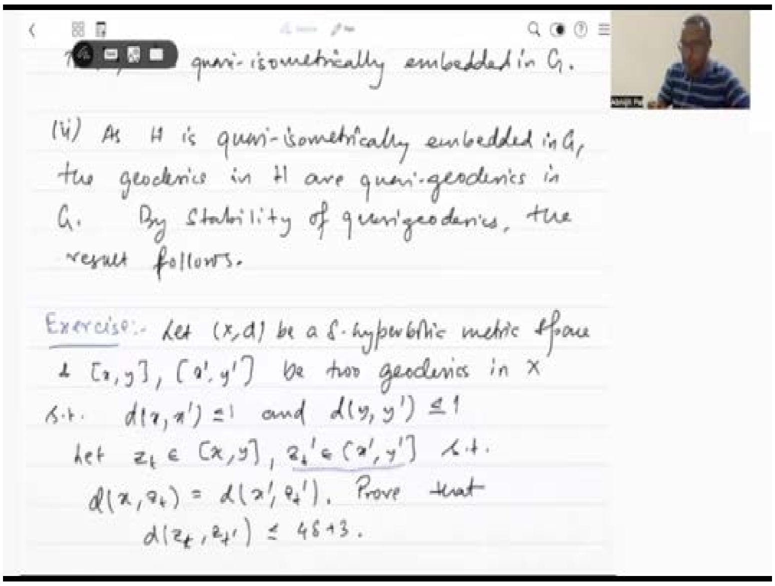

Now, for the second part of our proof, we aim to demonstrate that H is indeed a hyperbolic group. This conclusion follows from the stability of quasi-geodesics. Since H is quasi-isometrically embedded in G, it follows that geodesics in H will be quasi-geodesics in G.

By applying the stability of quasi-geodesics, you can readily prove that the triangles within H are slim. I will leave this as an exercise for you to explore further.

There's another exercise that states the following: Suppose we have two geodesics, x and y, and another pair x' and y', such that the distance between x and x' is less than or equal to 1, and the distance between y and y' is also less than or equal to 1.

Now, consider any point z_t on the geodesic joining x and y, as well as the point $z_{t'}$ on the geodesic connecting x' and y'. If the distance from x to z_t is the same as the distance from x' to $z_{t'}$, then it can be shown that the distance between z_t and $z_{t'}$ is less than or equal to $4\delta + 3$.

I will leave this as an exercise for you to work through. Alright, I will stop here.