An Introduction to Hyperbolic Geometry

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Lecture – 38

Cone Types and Infinite Order Elements in Hyperbolic Groups

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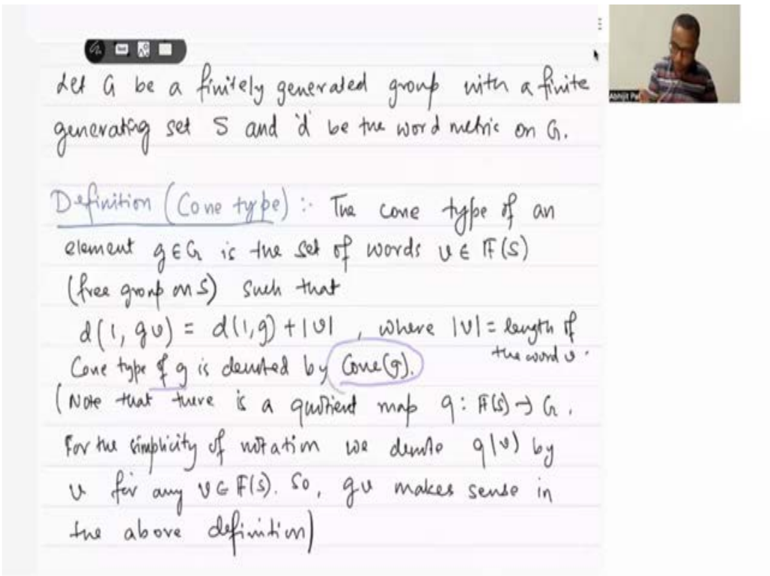

Hello everyone! In this lecture, we will demonstrate that an infinite hyperbolic group contains an element of infinite order. To establish this, we will define the concept of the cone type of an element within a group, and we will show that in a hyperbolic group, there are finitely many cone types. This will lead us to conclude that an infinite hyperbolic group must indeed have an element of infinite order.

Let's begin by considering a group that is finitely generated, with a finite generating set denoted by S, and let d represent the metric on G.

Now, let's define what we mean by cone type. The cone type of an element g is defined as the set of words v, where v belongs to the free group generated by S, such that the distance from 1 to gv is equal to the distance from 1 to g plus the length of the word v.

If I were to illustrate this, the path from 1 to g represents the geodesic connecting 1 and g. Additionally, the path from g to gv also represents the geodesic joining g and gv.

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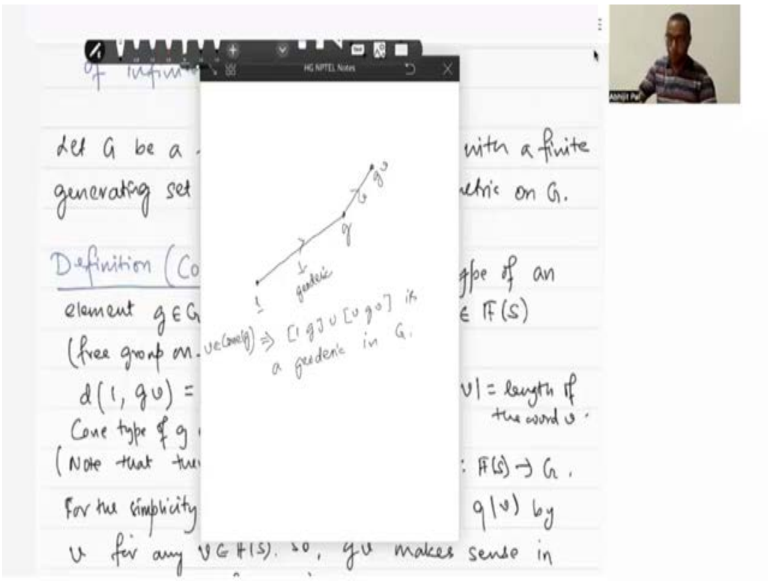

Now, when we consider the union of these two paths, the union of these two geodesics, we can say that if v belongs to the cone of g, then the concatenation of these two paths forms a geodesic in G. Therefore, this union, or the concatenation of these two paths, becomes a geodesic in G when v is part of the cone of g.

Let's denote the cone type of g by $\mathrm{Cone}(g)$.

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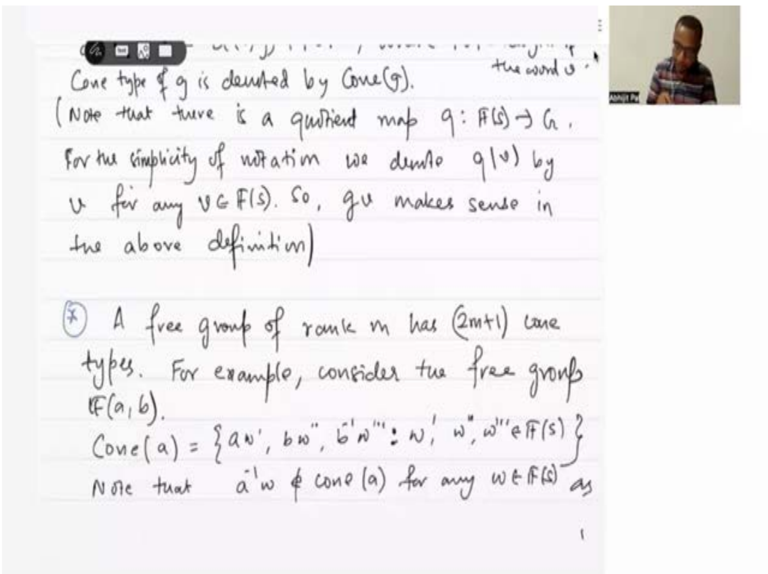

Now, let's observe that there exists a quotient map from the free group S to the group G. For the sake of simplicity in our notation, we will denote q(v) by v for any word v that belongs to the free group generated by S. This allows us to work with G in the context of the definitions we are using.

Now, let’s delve into an example. Consider a free group of rank m. It can be shown that this group possesses 2m + 1 cone types. For instance, if we take the free group generated by two elements, a and b, we can analyze the cone type associated with a. This cone type consists of the set of all words of the form a w', where w' belongs to the free group generated by S, v v w'' belongs to the free group generated by S, and $v^{-1}w'''$ also belongs to the free group generated by S.

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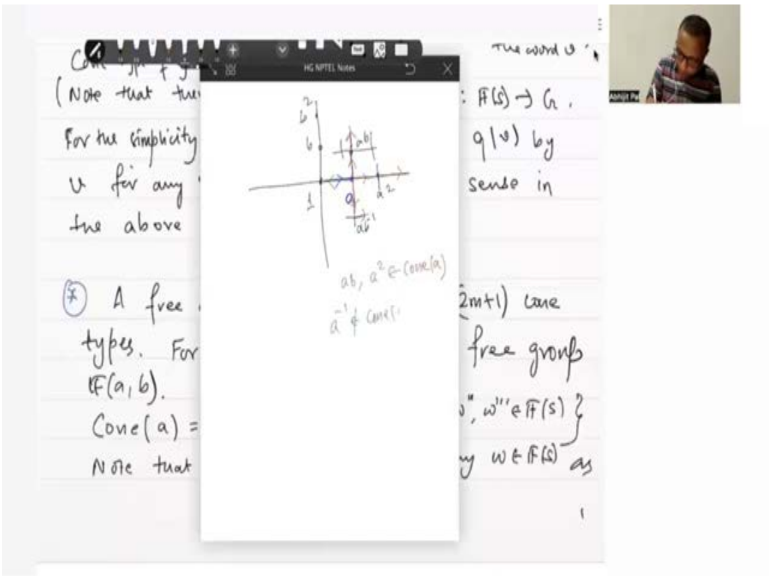

To visualize this concept, let’s first consider the Cayley graph of the free group generated by these two elements. This graph provides a representation of the structure of the group. Now, when we examine the element a, we want to understand what it means for an element to belong to the cone type of a.

We begin with a geodesic path from the identity element 1 to a. If we take the element a^2, which can be represented by a blue geodesic followed by a red one, we see that this sequence forms a geodesic. Essentially, the path that moves in the blue direction, followed by the red direction, remains a geodesic.

Consequently, the expression ab a^2 falls within the cone of a. The same principle applies in the opposite direction; if we again start with the blue path and follow it with the red path, this will also be a geodesic.

However, to identify elements that do not belong to the cone of a, consider the path where we first go from 1 to a and then return back to 1 using the edge corresponding to a^{-1}. This sequence demonstrates that a^{-1} does not belong to the cone of a.

So, any word of the form $aw'bw''b^{-1}w'''$ will reside within the cone of a. On the other hand, any word that begins with a^{-1} w does not belong to the cone of a.

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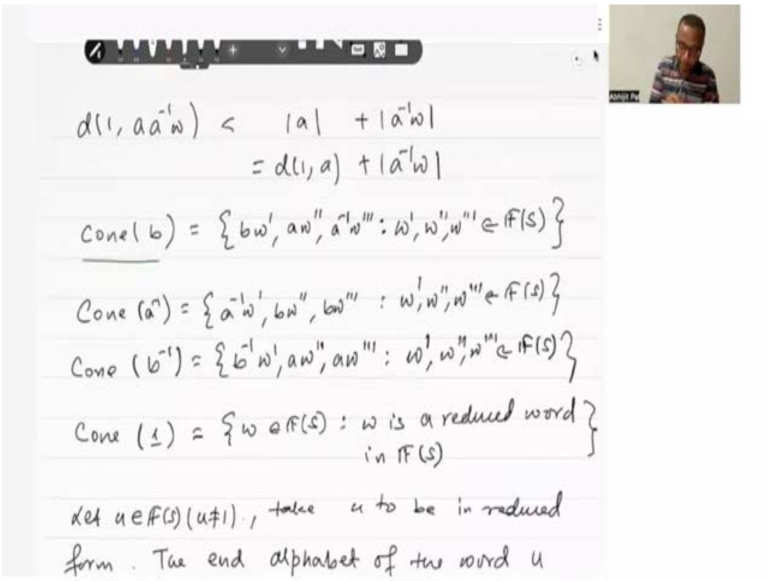

Let’s analyze the distance between the identity element 1 and the element $a^{-1}w$. This distance is strictly less than the sum of the lengths of a and $a^{-1}w$. In fact, the length of a is exactly equal to the distance from 1 to a plus the length of $a^{-1}w$. Consequently, this implies that $a^{-1}w$ does not belong to the cone of a.

In a similar manner, we can discuss the cones of b, a^{-1}, and b^{-1}, as well as the cone of the identity element. So, what does it mean to consider the cone of the identity element? It encompasses any word that is a reduced word in the free group generated by S. A reduced word is defined as one that does not contain any subwords of the form xx^{-1}; in other words, it must not include any instances where an element is immediately followed by its inverse within the word.

Now, let's consider any element u from this free group, ensuring that u is not equal to the identity and is in its reduced form. The final letter of this word u could be a, a^{-1}, b, or b^{-1}.

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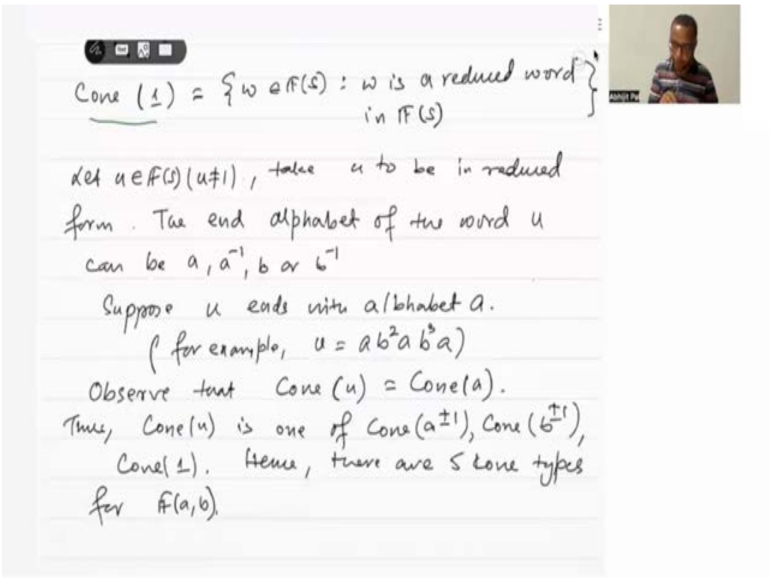

For instance, if u ends with the letter a, say $u = ab^2a^3$, we can observe that the cone of u is actually equal to the cone of a.

Therefore, we can conclude that the cone of any element u will be one of the following types: the cone of a, the cone of a^{-1}, the cone of b, the cone of b^{-1}, or the cone of the identity element. Consequently, we find that there are only five distinct cone types for this free group generated by a and b.

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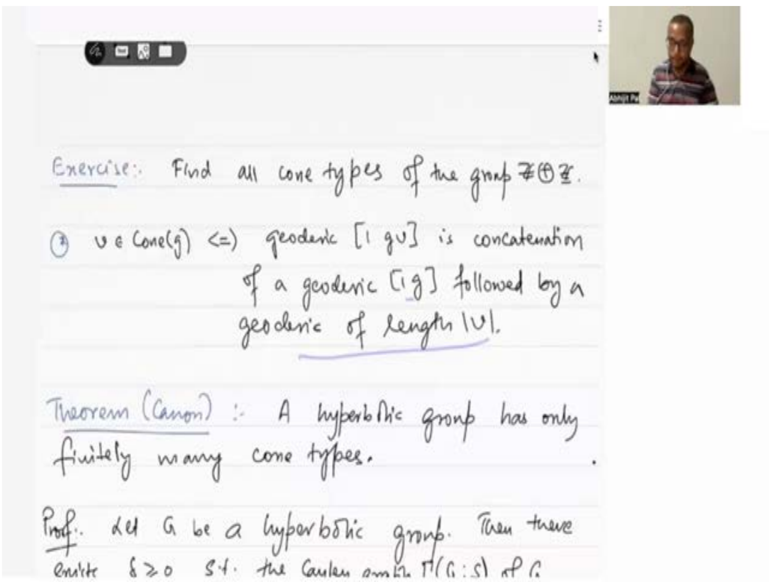

Now, let's consider an exercise: we need to identify all the cone types of the group $Z \times Z$. As

I mentioned earlier, an element v belongs to the cone of g if and only if the geodesic connecting the identity element 1 to gv can be expressed as the concatenation of two geodesics: one from 1 to g, followed by another geodesic of length equal to that of v.

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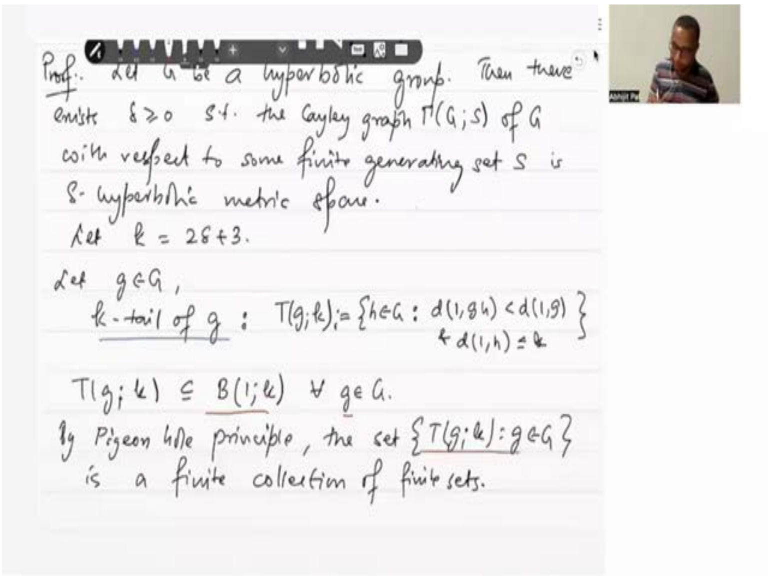

Now, let's discuss a theorem established by Cannon, which states that a hyperbolic group contains only finitely many cone types. To begin, we consider a hyperbolic group G. There exists a non-negative number δ such that the Cayley graph of G, with respect to a finite generating set S, is a δ-hyperbolic metric space. Next, we will define k to be equal to 2δ + 3.

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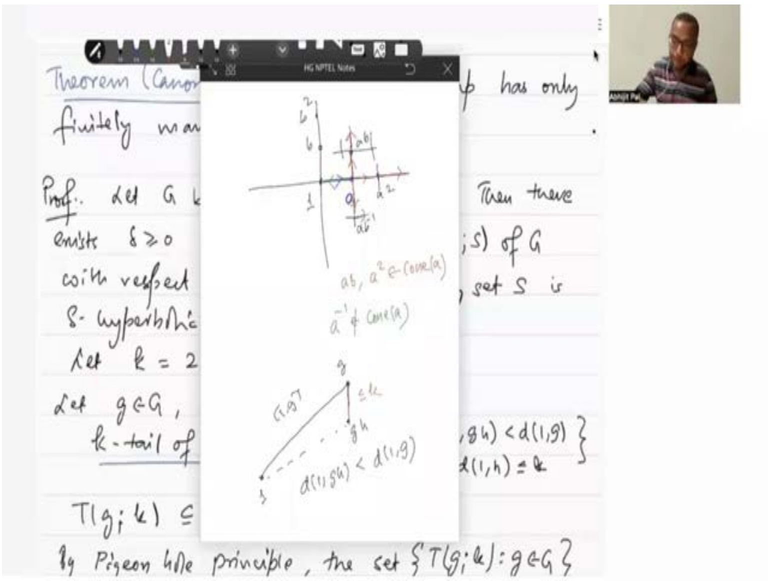

So, what are we going to do? We are going to define the k-tail of an element in the group G. The k-tail of an element g is defined as the set of all elements h belonging to G such that the distance from 1 to gh is less than the distance from 1 to g, and the distance from 1 to h is less than or equal to k. In essence, this tail is the opposite of the cone.

Let's visualize this with a diagram: here we have the geodesic joining 1 and g, as well as the geodesic connecting g and gh. Now, consider the distance between 1 and gh. If this distance is less than the distance between 1 and g, and the length of this geodesic is less than or equal to k, we will refer to h as being in the k-tail of the element g.

From the definition, it becomes evident that the k-tail of an element g is contained within the ball centered at 1 with a radius of k. This statement holds true for all elements g. Consequently, by the Pigeonhole Principle, we can conclude that this ball of radius k is a finite set, as we are working with a finitely generated group. Therefore, the collection of these tails forms a finite collection of finite sets.

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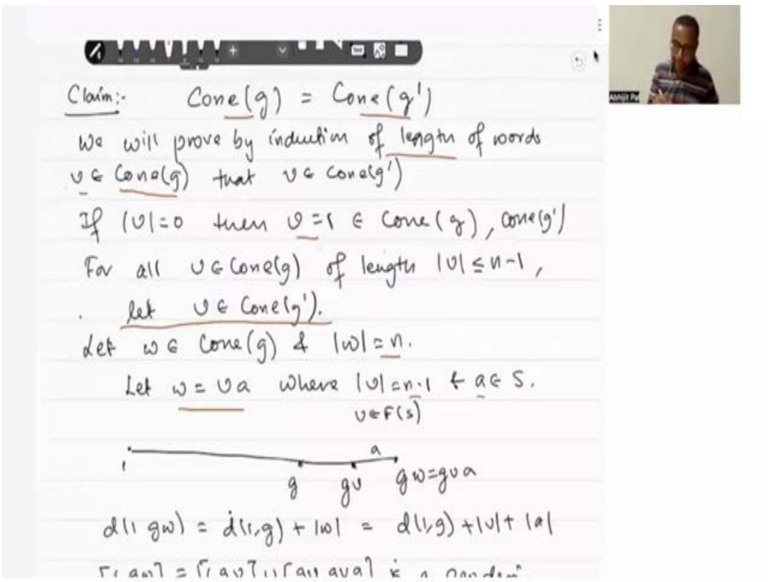

Thus, we can find two elements, g and g', such that the tail of g is equal to the tail of g'. Here, we assume that g belongs to an infinite hyperbolic group S. Given this, there exist g and g' such that their tails are identical. Our goal now is to show that the cone of g is equal to the cone of g'. If we can establish this equivalence, we will have demonstrated that there exist finitely many cone types.

To achieve this, we will use induction on the length of the words of an element v that belongs to the cone of g. We begin with an element v in the cone of g and apply induction to prove that v also belongs to the cone of g'.

First, suppose that the length of v is 0. In this case, v represents the identity element, which clearly belongs to both the cone of g and the cone of g'.

Next, we assume that for all v in the cone of g with a length less than or equal to n - 1, these elements also belong to the cone of g'. This assumption serves as our induction hypothesis.

Now, let us take an element w in the cone of g such that the length of w is equal to n. According to our induction hypothesis, all elements b with lengths less than or equal to n - 1 belong to the cone of g'.

We will demonstrate that w also belongs to the cone of g'. Let's express w as w = pa, where the length of b is n - 1 and a is a generator. Thus, we can rewrite w as w = ba, where b is a word of length n - 1 and a is a generator.

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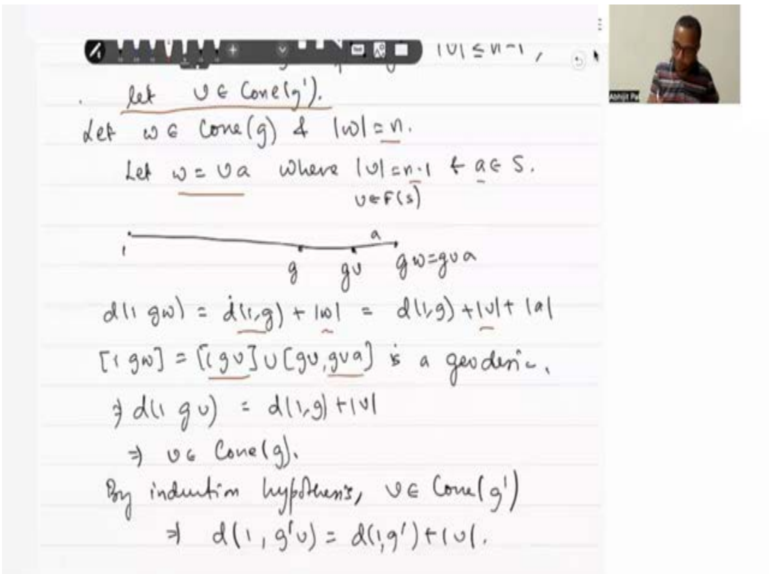

Now, since w belongs to the cone of g, we can express the distance from 1 to gw as follows: it is equal to the distance from 1 to g plus the length of w. This can be further broken down into the distance from 1 to g plus the lengths of b and a, which yields the equation:

$$\text{Distance}(1, gw) = \text{Distance}(1, g) + \text{Length}(w)$$
$$= \text{Distance}(1, g) + \text{Length}(b) + \text{Length}(a) = 1.$$

Additionally, the geodesic that connects 1 to gw is the concatenation of the geodesic from 1 to gv and the geodesic that joins gv and gva. Since the segment from 1 to gv is also a geodesic, we find that:

$$\text{Distance}(1, gv) = \text{Distance}(1, g) + \text{Length}(v).$$

This implies that v must belong to the cone of g.

We began with an element w that is contained in the cone of g, expressed as w = va, where the length of v is equal to n - 1 and v is also a member of the cone of g. By our induction hypothesis, this means that v belongs to the cone of g'. Thus, we can conclude that the distance from 1 to g'v can similarly be expressed as:

$$\text{Distance}(1, g'v) = \text{Distance}(1, g') + \text{Length}(v).$$

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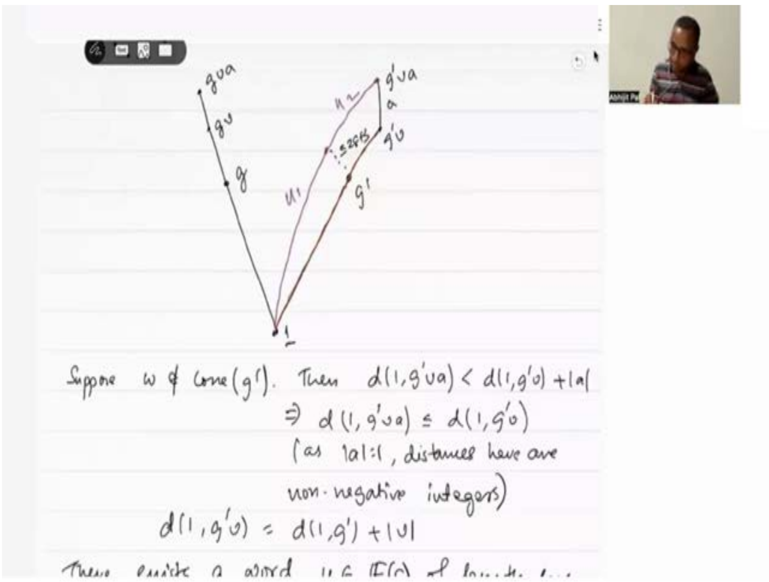

The path from 1 to g'v will indeed form a geodesic. Now, let's consider the scenario where w does not belong to the cone of g'. Our goal is to demonstrate that w must actually belong to the cone of g'.

In this case, the distance from 1 to g'v can be expressed as being strictly less than the distance from 1 to g'v plus the length of a. Mathematically, we write:

$$\text{Distance}(1, g'v) < \text{Distance}(1, g'v) + \text{Length}(a).$$

This means that the distance from 1 to g'v is less than the total length of the geodesic plus the length of the edge a, which is equal to 1. Since the absolute value of a is 1, we have:

$$\text{Distance}(1, g'va) \leq \text{Distance}(1, g'v).$$

This inequality holds true because the distances involved are non-negative integers. Thus, it follows that w must indeed belong to the cone of g'.

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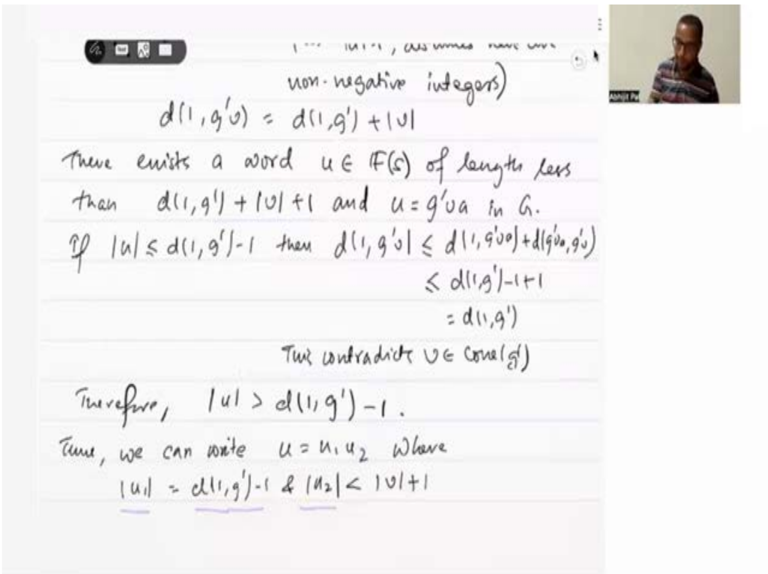

We also know that the distance from 1 to g'v can be expressed as the sum of the distance from 1 to g' plus the length of v. From this, it follows that there exists a word u in the free group generated by S such that the length of u is less than the distance from 1 to g' plus the length of v plus 1. This word u represents the element g'va in the group G.

Now, let’s analyze the situation. If the length of u is less than or equal to the distance from 1 to g' minus 1, then, by the triangle inequality, the distance from 1 to g'v would be less than or equal to the distance from 1 to g'. This would contradict our assumption that v belongs to the cone of g'. According to our induction hypothesis, v indeed belongs to this cone, which implies that if the length of u is less than or equal to the distance from 1 to g' minus 1, it would lead to a contradiction regarding b belonging to the cone of g'.

Therefore, we conclude that the length of u must be strictly greater than the distance from 1 to g' minus 1. This allows us to express u as the concatenation of two words, u_1 and u_2, where the length of u_1 equals the distance from 1 to g' minus 1 and the length of u_2 is less than the length of v plus 1.

It is important to note that u represents the element g'va in the group G. Hence, the total length of u must be strictly less than the distance from 1 to g' plus the length of b plus 1. This provides the basis for our conclusion: we can indeed split u into u_1 and u_2 such that the length of u_1 equals the distance from 1 to g' minus 1 and the length of u_2 is strictly less than the length of v plus 1.

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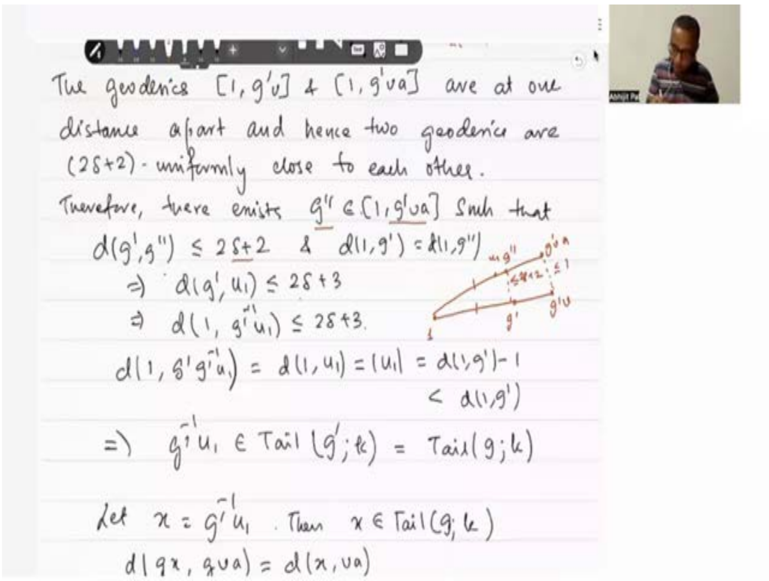

Now, let’s take note of the geodesics from 1 to g'v and from 1 to g'va. These two paths are just one unit apart, meaning they are $2\delta + 2$ uniformly close to each other. When we say "uniformly close," we mean that if we designate the path from 1 to g'v as α and the path from 1 to g'va as β, we have previously established that the distance between the points $\alpha(t)$ and $\beta(t)$ is less than or equal to $2\delta + 2$.

At one end of both paths, the starting point is 1, while the other endpoints are just one unit apart from each other. This means that the distance between $\alpha(t)$ and $\beta(t)$ remains constrained to be less than or equal to $2\delta + 2$. This definition of being uniformly close is crucial.

Because of this uniform proximity, we can assert that there exists a point g'' on the geodesic connecting 1 to g'va such that the distance between g' and g'' is also less than or equal to 2δ + 2. Additionally, the distances from 1 to both g' and g'' are equal.

I have illustrated this scenario: for the point g', there exists g'' such that the length of the geodesic from 1 to g' is identical to that of the geodesic from 1 to g'', with the distance being less than or equal to 2δ + 2.

Now, regarding the point u_1, which will be located somewhere along this path, it is defined such that the length of u_1 equals the distance from 1 to g' minus 1. The word u represents the element g'pa, indicating that u_1 is positioned accordingly.

By applying the triangle inequality, we can determine that the distance between g' and u_1 will be equal to 2δ + 3. Thus, we conclude that the distance from 1 to $g'^{-1}u_1$ is less than or equal to 2δ + 3, which is significant because it corresponds exactly to our defined value of k.

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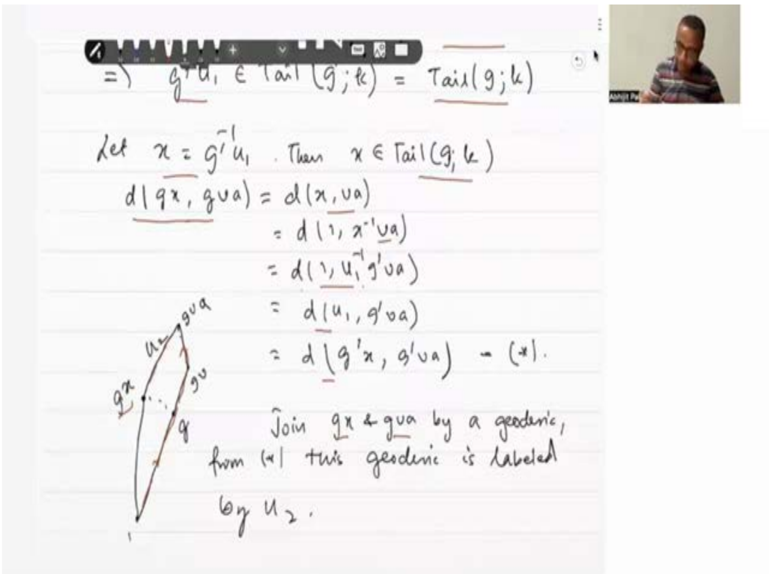

Additionally, the distance from 1 to g' g'^{-1} u_1 is precisely equal to the distance from 1 to u_1, which corresponds to the length of u_1. This distance is also equal to the distance from 1 to g' minus 1. This choice of u_1 is significant because it is strictly less than the distance from 1 to g'. This establishes that g'^{-1} u_1 indeed belongs to the tail of g'.

Furthermore, we note that this tail of g' is equivalent to the k-tail of g. Consequently, the element $x = g'^{-1}u_1$ falls within the k-tail of g.

Now, if we consider the element g x, the distance between g x and g v_a can be expressed as the distance between x and v_a because left translation in this group operates isometrically. This can be further broken down to the distance from 1 to x^{-1} v_a when we substitute x in the equation.

From this perspective, we can say that the distance from 1 to u_1^{-1} g' v_a matches the distance we calculated previously. In essence, this distance is equivalent to the distance between u_1 and g' v_a.

We have identified u_1 as exactly equal to g' x, which means we can rewrite this as the distance between g' x and g' v_a. Thus, we have established that the distance between g x and g v_a is the same as the distance between g' x and g' v_a.

Next, we will connect g x and g v_a with a geodesic, which we will label as u_2. So, u_1 corresponds to one geodesic, while u_2 represents this new geodesic. Together, u_1 and u_2 constitute the geodesic joining 1 to g' v_a.

In this scenario, we have traversed from the geodesic connecting 1 to g and from g to v_a. The concatenation of these paths forms a geodesic, as we have taken g x into account. Therefore, the distance from g x to g v_a is identical to the distance between g' x and g' v_a, which corresponds to the length of u_2. Thus, we can appropriately label this geodesic as u_2.

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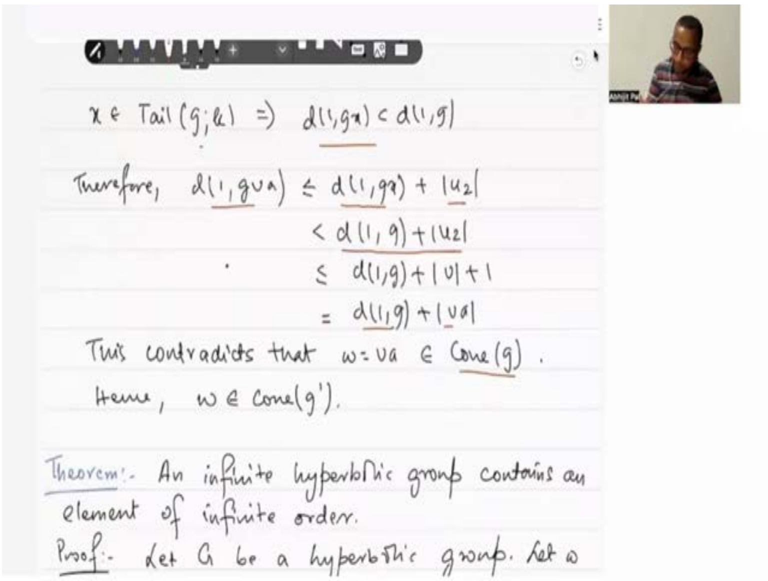

Now, let's revisit our variable x, which was defined as g'^{-1} u_1. Since x belongs to the tail of g with respect to k, it follows that the distance from 1 to g x is strictly less than the distance from

1 to g. Consequently, the distance from 1 to g v_a can be expressed as being less than or equal to the distance from 1 to g x plus the length of u_2.

This leads us to conclude that the distance from 1 to g x is indeed less than the distance from 1 to g plus the length of u_2. Now, let's take a look at our diagram. We see a path that connects 1 to g v_a, and the length of this path is the sum of the distance from 1 to g x and the length of u_2. Therefore, the distance from 1 to g v_a must be less than or equal to the length of that path, which we can express as:

$$\text{distance}(1, gv_a) \leq \text{distance}(1, gx) + \text{length}(u_2).$$

Since we have established that the distance from 1 to g x is strictly less than the distance from 1 to g plus the length of u_2, we can further refine our analysis. Given that we selected u_2 such that its length is less than or equal to $\text{length}(b) + 1$, we can thus state that:

$$\text{distance}(1, gv_a) < \text{distance}(1, g) + \text{length}(b) + 1.$$

Since the expression for $\text{length}(b) + 1$ corresponds to the length of the word v_a, we conclude:

$$\text{distance}(1, gv_a) < \text{distance}(1, g) + \text{length}(v_a).$$

This leads us to a contradiction because we initially posited that w = v_a belongs to the cone of g.

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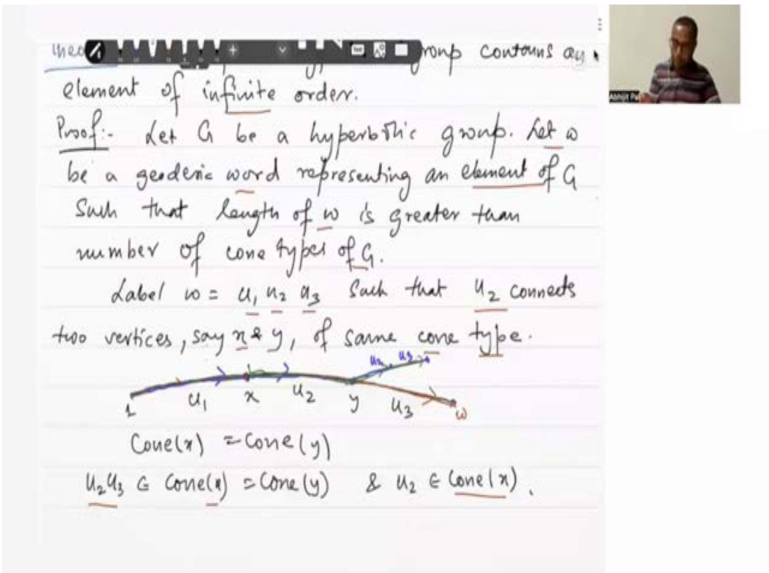

Thus, we must assert that w indeed belongs to the cone of g'. In conclusion, we have successfully demonstrated this theorem: there exists a finite number of cone types within an infinite hyperbolic group.

Now, let us consider an infinite hyperbolic group. We aim to prove that such a group contains an element of infinite order. Let G be an infinite hyperbolic group, and let w be a geodesic word that represents an element of G, with the length of w exceeding the number of cone types in G. This is always feasible because G is an infinite group, allowing us to construct words of significant length.

Since the length of w is greater than the number of cone types in G, we can express w in a specific manner. In particular, w can be written as $w = u_1 u_2 u_3$, where u_2 connects two vertices, say x and y, which share the same cone type.

To visualize this, we can represent the geodesic word w with a diagram. There exist two vertices on this geodesic, x and y, such that the cone type of x is the same as that of y. In this representation, we label the geodesic from 1 to x as u_1 and the geodesic from x to w as u_3.

Now, observe that the words u_2 and u_3 belong to the cone of x because the entire construction represents a geodesic. Therefore, u_2 and u_3 are elements of the cone of x, and since the cone of x is precisely equal to the cone of y, it follows that u_2 is also in the cone of x.

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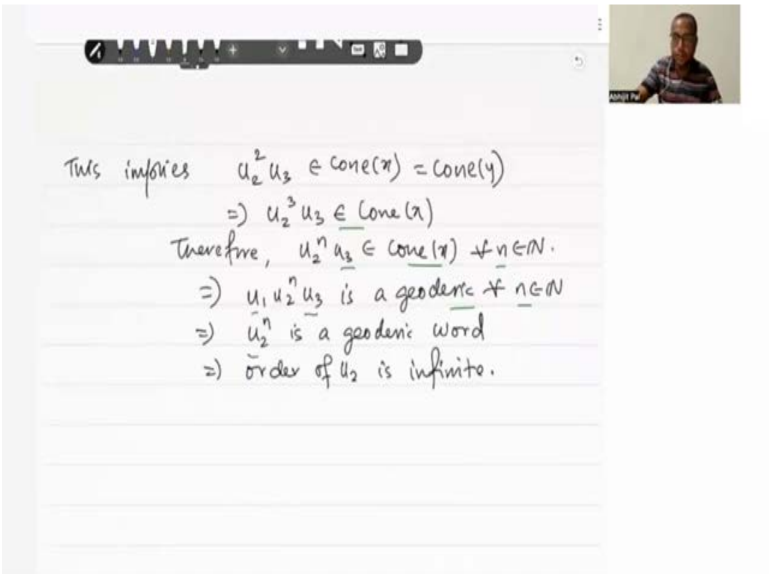

This leads us to conclude that $u_2^2 u_3$ must also belong to the cone of x. Since u_2 and u_3 are also in the cone of y, it implies that if we consider the geodesic word u_2 u_3, it lies within the cone of y as well. Therefore, the green path from 1 to y traversing along this geodesic will also be a geodesic. This indicates that u_2^2 u_3 is a geodesic and indeed resides within the cone of x.

In summary, we have demonstrated that the structure of the geodesic word allows us to derive paths that affirm the existence of an element of infinite order within the infinite hyperbolic group.

Once again, we note that the cone of x is equal to the cone of y. Utilizing this observation, we can similarly demonstrate that u_2^3 u_3 also belongs to the cone of x. This process can be repeated, leading us to conclude that for all natural numbers n, the expression u_2^n u_3 resides within the cone of x.

This establishes that u_1 u_2^n u_3 forms a geodesic for all n, and importantly, u_2^n represents a geodesic word. Consequently, it cannot correspond to the identity element. Thus, we deduce that the order of u_2 is indeed infinite.

In summary, we have successfully proven this theorem: in an infinite hyperbolic group, there exists an element of infinite order. I will conclude my presentation here.