An Introduction to Hyperbolic Geometry

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Lecture – 37

An Introduction to Hyperbolic Groups and Their Finite Presentations

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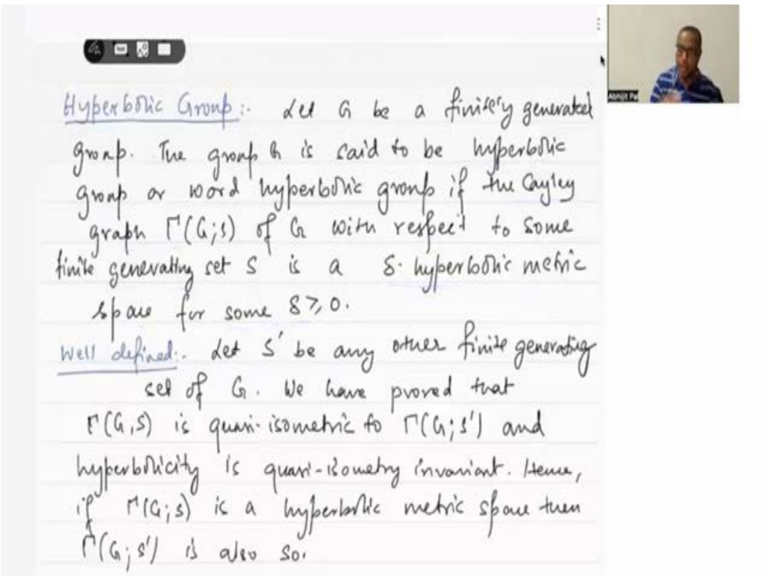

Hello! In this lecture, we will delve into the concept of hyperbolic groups and proceed to prove that every hyperbolic group is finitely presented. Let's begin by defining what a hyperbolic group is.

A hyperbolic group, or more specifically, a word hyperbolic group, is a finitely generated group with a particular property: the Cayley graph of this group, with respect to some finite generating set S, forms a δ-hyperbolic metric space for some non-negative number δ.

In other words, given any finitely generated group G, one can construct its Cayley graph based on the generating set S. If this Cayley graph behaves as a hyperbolic metric space (as defined in geometric group theory), then the group G is classified as a hyperbolic group.

Now, an important question arises: Is this definition well-defined?

There can be many generating sets for the finitely generated group G. Now, suppose S' is another finite generating set for this group G. We have already established that the Cayley graph of G with respect to S is quasi-isometric to the Cayley graph of G with respect to the generating set S'. In fact, in our previous lecture, we demonstrated that hyperbolicity is an invariant under quasi-isometry.

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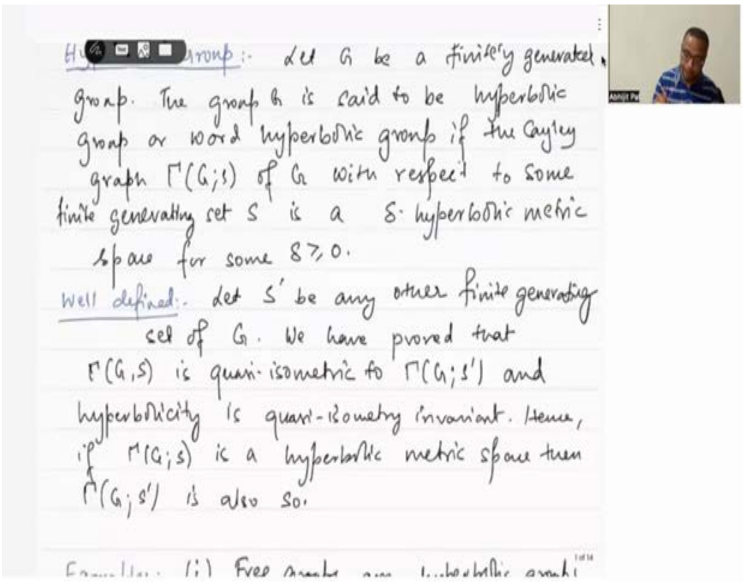

Therefore, if the Cayley graph Γ(G, S) is a hyperbolic metric space, then the Cayley graph Γ(G, S') will also be hyperbolic. This proves that the definition of hyperbolicity is well-defined in the sense that, regardless of which finite generating set we choose, the corresponding Cayley graphs will all be hyperbolic metric spaces.

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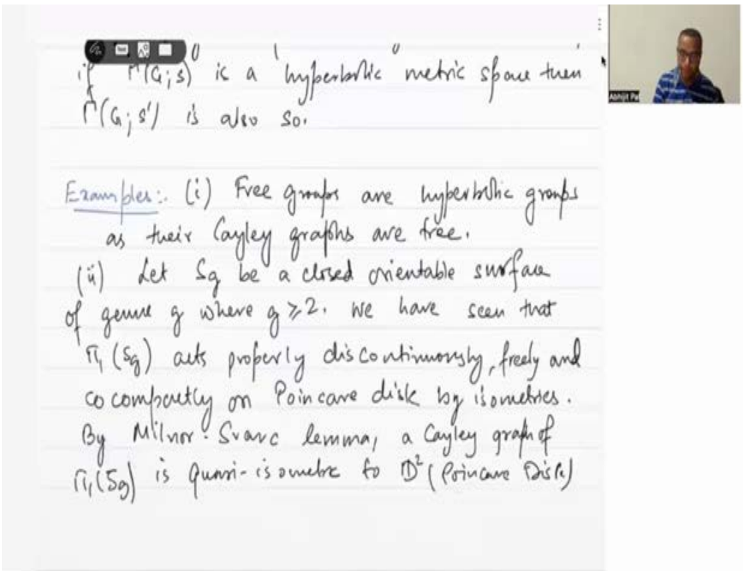

For instance, consider free groups of any rank. These free groups are hyperbolic because their Cayley graphs exhibit free structure. Now, let's take the example of a closed orientable surface of genus g, where $g \geq 2$. We know that the fundamental group of such a surface acts properly discontinuously, freely, and co-compactly by isometries on the unit disk model of the hyperbolic plane. That is the key observation here.

By the Milnor-Švarc Lemma, if we consider the Cayley graph of this fundamental group with respect to some finite generating set, it will be quasi-isometric to the Poincaré disk, which is itself a hyperbolic metric space. Since hyperbolicity is a quasi-isometry invariant, the Cayley graph of this fundamental group will also be a hyperbolic metric space.

Thus, the fundamental group of a surface of genus g, where $g \geq 2$, is indeed a hyperbolic group.

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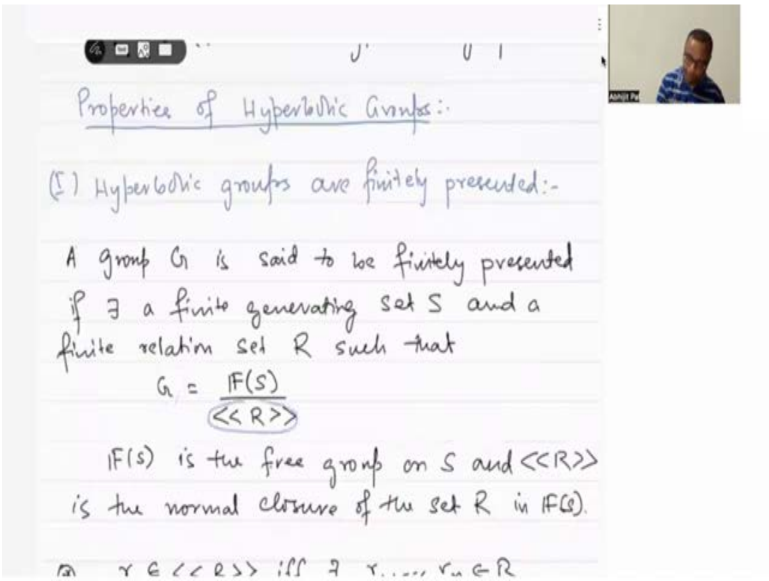

Next, we will prove that hyperbolic groups are finitely presented. Before diving into the proof, let's recall the definition of a finitely presented group. A group is said to be finitely presented if there exists a finite generating set S and a finite set of relations R such that the group can be expressed as a quotient. Specifically, the group is the quotient of the free group generated by S, and the denominator consists of the normal closure of the set R, where R is a finite set.

In other words, this means that the group can be described with a finite number of generators and a finite number of relations among those generators, which greatly simplifies its structure and the way it can be studied or manipulated algebraically.

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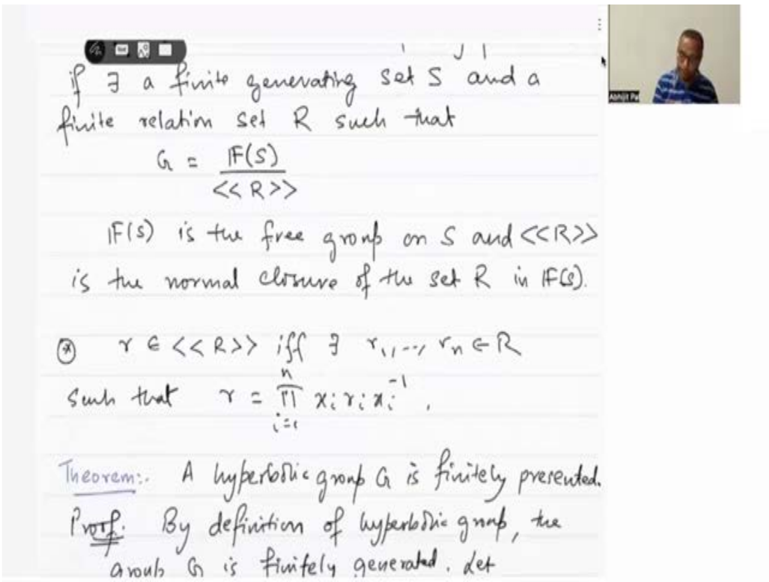

Now, note that if we take any element R belonging to the normal closure containing r, there will exist finitely many elements, say $r_1, r_2, \ldots, r_n$, such that R can be expressed as the product of conjugates of these r_i. In other words, $R = \prod(x_i r_i x_i^{-1})$, where i runs from 1 to n.

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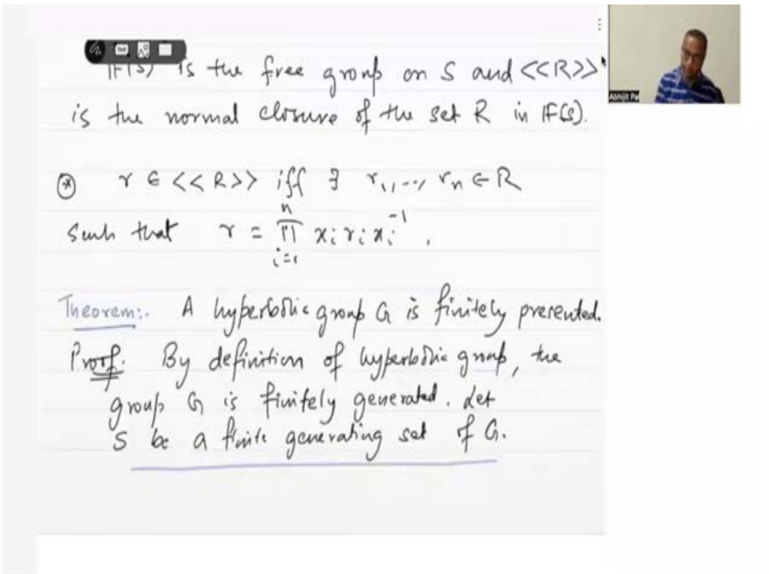

Conversely, if an element can be written as the product of such conjugates, then R must belong to the normal closure containing the set R. This is a straightforward observation, something that is usually encountered while studying group theory and normal closures.

Now, let's state the theorem: A hyperbolic group is finitely presented.

By the very definition of a hyperbolic group, the group G is finitely generated. Let S be a finite generating set of G. We will build upon this foundational aspect as we move forward in the proof. This key fact ensures that the group structure is well-defined and manageable, paving the way for further analysis.

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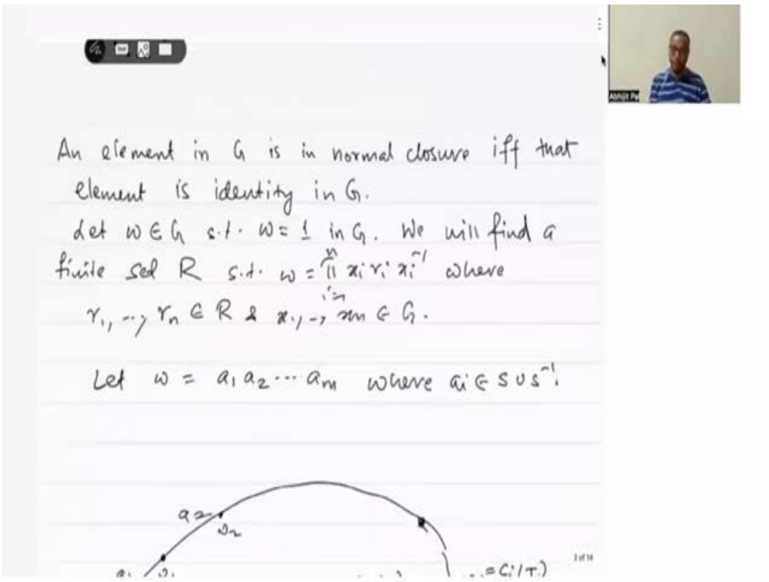

First of all, let's note that an element in G belongs to the normal closure only if it is the identity element in G. So, suppose we take an element w which belongs to the denominator, i.e., the normal closure. In that case, that element must be the identity element in G. The converse also holds true, if an element is the identity element in G, then it belongs to the normal closure.

Therefore, there must exist elements $r_1, r_2, \dots, r_n$ in R such that this w can be expressed as the product of conjugates $x_i r_i x_i^{-1}$. Now, let's express w as $w = a_1 a_2 \dots a_m$, where each a_i belongs to the generating set S, meaning either $a_i \in S$ or $a_i^{-1} \in S$. This implies that each a_i is either an element of S or its inverse. Therefore, w is represented as a product of elements from $S \cup S^{-1}$.

Since w = 1, it represents a loop. Let's denote this loop as ν, which corresponds to w. Now, w starts from the identity element, 1, and forms an edge in the Cayley graph. The first vertex in this loop is v_1, and there is an edge between 1 and v_1, labeled by a_1. The next vertex is v_2, with an edge between v_1 and v_2 labeled by a_2, and this pattern continues.

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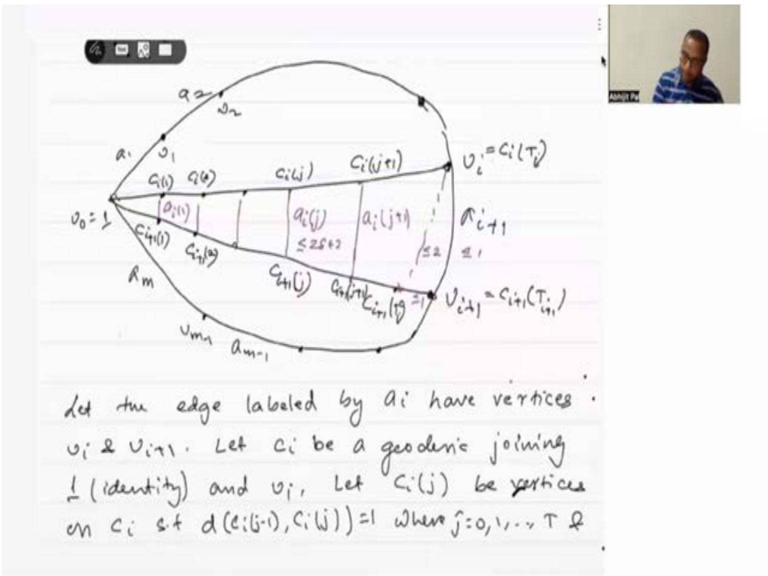

Each edge between vertices v_i and v_{i+1} is labeled by a_{i+1}. Now, what we do next is, for each vertex v_i, we take a geodesic c_i that joins the identity element and v_i. This means that c_i is the geodesic joining 1 and a_i.

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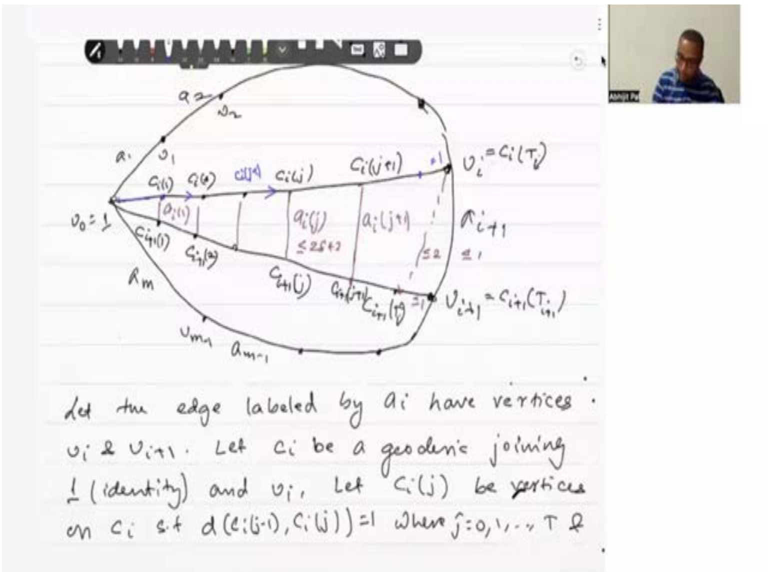

So, let's proceed with this construction. For every index i, we obtain a corresponding geodesic. Now, note that the distance between any two consecutive vertices, v_i and v_{i+1}, is at most 1, since the length of the edge between them is exactly 1. This is an important observation.

Next, let us introduce the notation c_{ij}, where c_{ij} represents vertices on the geodesic c_i, and the distance between consecutive vertices $c_{i(j-1)}$ and c_{ij} is exactly 1. Thus, we can split the geodesic c_i into segments, as follows:

- Start by taking c_{i1}, such that the distance between the identity element and c_{i1} is precisely 1.
- Then, for each subsequent point, the distance between $c_{i(j-1)}$ and c_{ij} remains 1. Eventually, for the last point, say $c_{i(ji)}$, we arrive at vertex v_i.

Now, observe that for each vertex v_i, the distance between the identity element and a_i is a natural number, which means that the distance between consecutive vertices $c_{i(j-1)}$ and c_{ij} is always exactly 1.

Thus, for all i, you follow this same construction, and you'll end up with a clear picture of the structure. To complete the picture, join c_{i1} and $c_{(i+1)1}$ by geodesics. These are denoted by c_{ij} and $c_{(i+1)j}$, and we label each geodesic with a_{ij}.

This setup creates a well-structured framework of geodesics and vertices, where each edge is labeled accordingly, and the distances between the points are consistently maintained as per the geodesic properties.

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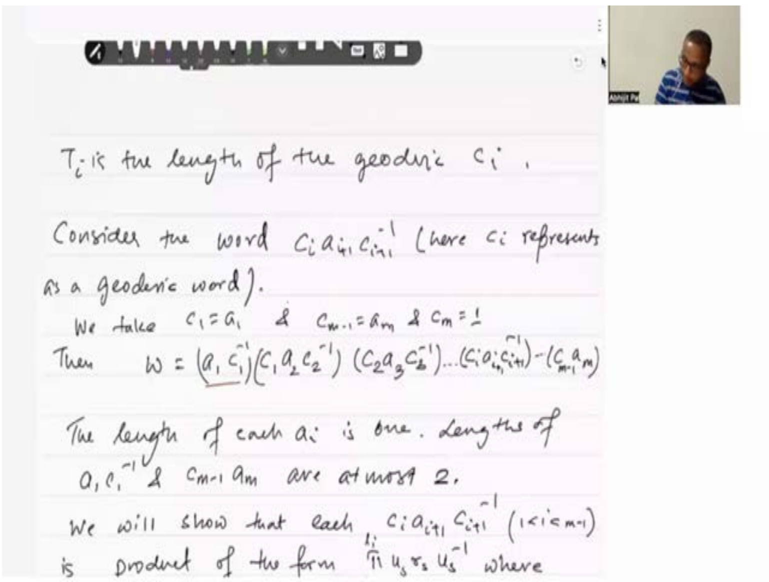

Now, let's focus on constructing this particular word. Consider the word $c_i a_{i+1} c_{i+1}^{-1}$, where c_i represents a geodesic word. What we do here is first trace along c_i, then move to a_{i+1}, and finally return via c_{i+1}^{-1}. This forms a loop, represented as $c_i a_{i+1} c_{i+1}^{-1}$, within the Cayley graph.

Next, define $c_1 = a_1, c_{m-1} = a_m$, and $c_m = 1$. Now, notice that the word w can be expressed as:

$$w = a_1 c_1^{-1} c_1 a_2 c_2^{-1} c_2 a_3 c_3^{-1} \cdots c_i a_{i+1} c_{i+1}^{-1} \cdots c_{m-1} a_m$$

What have we done here? We started with this loop, w, and we deconstructed it into a product of smaller sub-loops. Each sub-loop follows the pattern: proceed along c_{i-1}, then traverse via a_i, and return along c_i. Essentially, w is the product of these smaller loops.

There are some minor variations at the beginning and end of the word. At the starting point, the loop begins as $a_1 c_1^{-1}$, and at the end, it closes with $c_{m-1} a_m$. This gives us a clear representation of how the loop w is constructed and decomposed into smaller geodesic segments within the Cayley graph.

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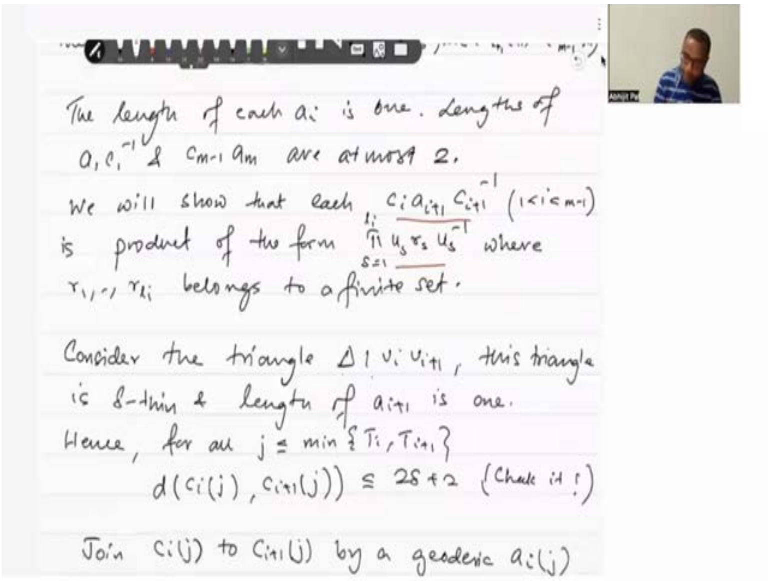

Now, let's analyze the length of each a_i. Since the length of each a_i is 1, we can observe that the total length of the loops $a_1 c_1^{-1}$ and c_{m-1} a_m will be at most 2. This is because c_1 corresponds to a_1 and c_{m-1} corresponds to a_m, leading to the conclusion that their combined length cannot exceed 2.

Next, we aim to demonstrate that each of these loops can be expressed as products of the form $u_s r_s u_s^{-1}$. Here, u_s and r_s belong to finite sets, meaning that $r_1, r_2, \dots, r_l$ belong to a finite collection. If we can establish this relationship, then the word w can be represented as a product of finitely many conjugates. Consequently, this means that the loop will be finitely presented.

Our next goal is to prove that each of these sub-loops can indeed be expressed as a finite product of these conjugates. To do this, we will examine the triangle formed by the points 1, T_i, and T_{i+1}.

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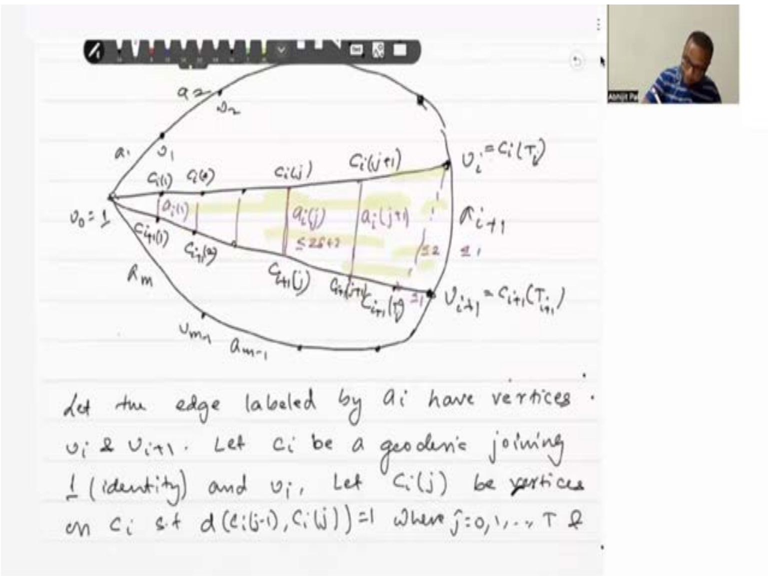

Now, let's take a closer look at this triangle; this is the triangle we will be focusing on this time.

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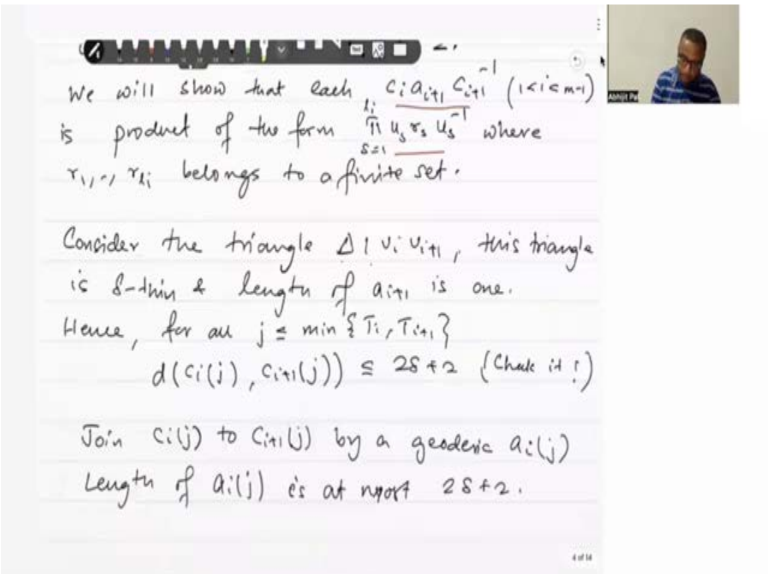

This triangle is denoted as Δ_n, and the length of a_{i+1} is equal to 1. Now, let's discuss T_i; T_i represents the length of the geodesic c_i, while T_{i+1} corresponds to the length of the geodesic a_{i+1}.

It can be established that for $j \leq \min(T_i, T_{i+1})$, the distance between the points c_{ij} and c_{i+1j} is at most 2δ + 2. This conclusion follows from the observations we've made.

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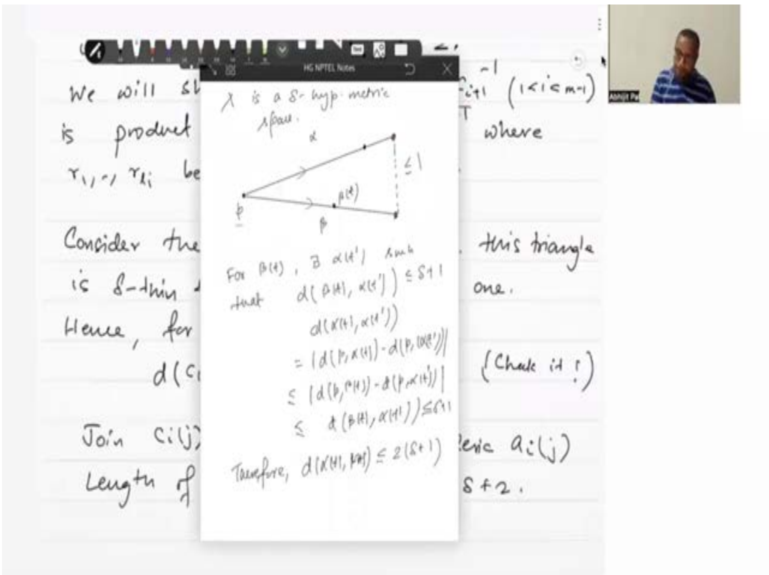

Let us begin with the concept of a δ-hyperbolic metric space. We'll denote our space as X and consider two geodesics, α and β, which have endpoints that are at most a distance of 1 apart. Both geodesics, α and β, originate from the same point P.

Now, if we take a point β_t along the geodesic β, there exists a point along α such that the distance between β_t and this point on α is less than or equal to δ. Assuming this point is indeed on α, the distance from β_t to the endpoint of α will be at most δ + 1. Consequently, we can find some t' such that the distance between $\alpha_{t'}$ and β_t is also at most δ + 1.

Applying the triangle inequality, we can then show that the distance between α_t and $\alpha_{t'}$ will be at most δ + 1. Furthermore, if we take t for α_t and t for β_t, we can deduce that the distance between α_t and β_t is less than or equal to 2δ + 1.

Using this information, we can establish that the distance between the points c_{ij} and c_{i+1j} is less than or equal to 2δ + 1. As previously mentioned, we can join c_{ij} and c_{i+1j} with the geodesic a_{ij}. This geodesic, labeled a_{ij}, will have a length of at most 2δ + 2.

Now, let us consider the words x_{ij}, which are defined as $x_{ij} = c_{ij}^{-1} c_{i(j+1)}$. Next, we define r_{ij} as follows:

$$r_{ij} = x_{ij} \cdot a_{i(j+1)} \cdot x_{(i+1)j}^{-1} \cdot a_{ij}^{-1}.$$

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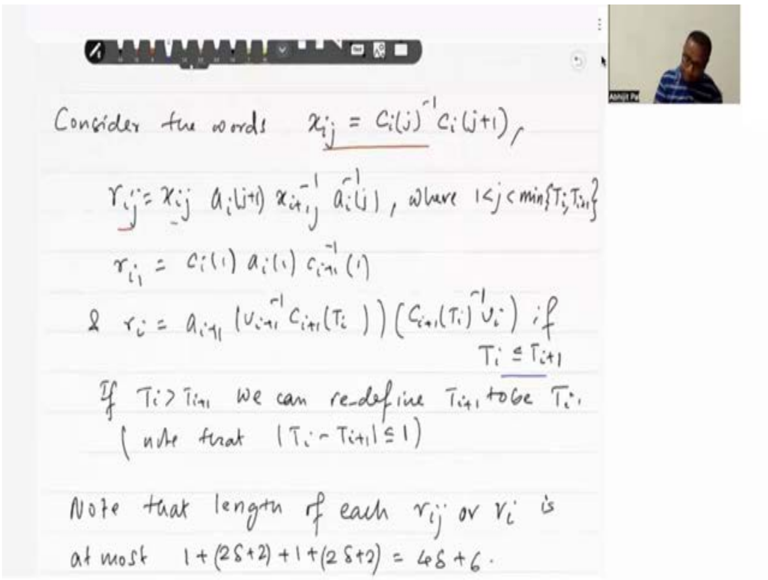

To visualize this, let's refer back to our diagram. The geodesic labeled a_{ij} starts from a specific point, followed by traveling along x_{ij}. What does r_{ij} represent? Essentially, it begins at this point, traverses along x_{ij}, continues along $a_{i(j+1)}$, then follows the geodesic, and finally returns via a_{ij}^{-1}. Thus, r_{ij} describes a loop characterized by:

$$r_{ij} = x_{ij} \cdot a_{i(j+1)} \cdot x_{(i+1)j}^{-1} \cdot a_{ij}^{-1},$$

where j is constrained between 1 and the minimum of T_i and T_{i+1}. For the case of r_{i1}, we find it simplifies to the expression for $c_{i1} \cdot a_{i1} \cdot c_{(i+1)1}^{-1}$.

Now, let's consider the expression for r_i:

$$r_i = a_{i+1} \cdot T_{i+1}^{-1} \cdot c_{(i+1)} \cdot T_i,$$

multiplied by $c_{(i+1)} \cdot T_i^{-1} \cdot T_i$. This holds true under the condition that T_i is less than or equal to T_{i+1}.

Now, let us note that the length of each r_{ij} or r_i is bounded by the expression $1 + 2\delta + 2 + 1 + 2\delta + 2$. To analyze this further, let's refer to our diagram for r_{ij}. The length of x_{ij} is equal to 1, while the length of the geodesic $a_{i(j+1)}$ is at most $\delta + 2$. The length of the geodesic c_{ij} is exactly equal to 1, and similarly, the length of the geodesic a_{ij} is also at most $\delta + 2$.

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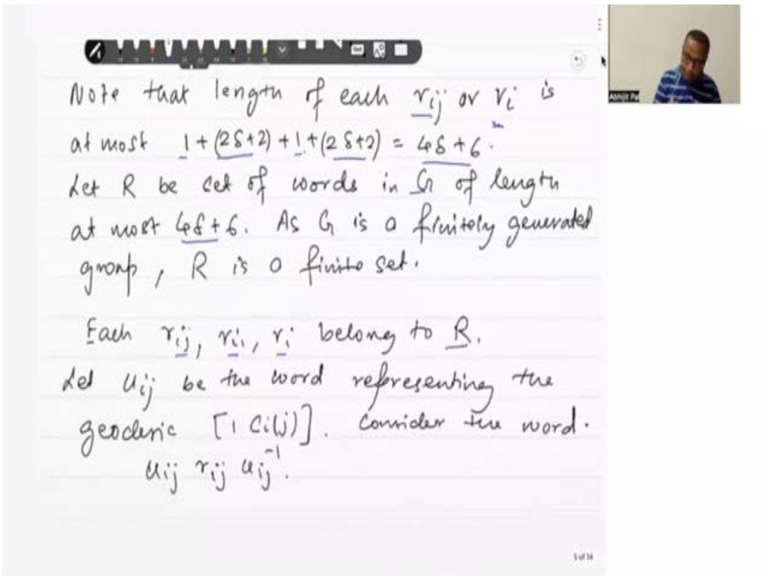

Putting all of this together, we can conclude that the length of r_{ij} is at most:

$$4\delta + 6.$$

By applying a similar reasoning, we find that the length of r_i, which appears at the end, will also be at most 4δ + 6.

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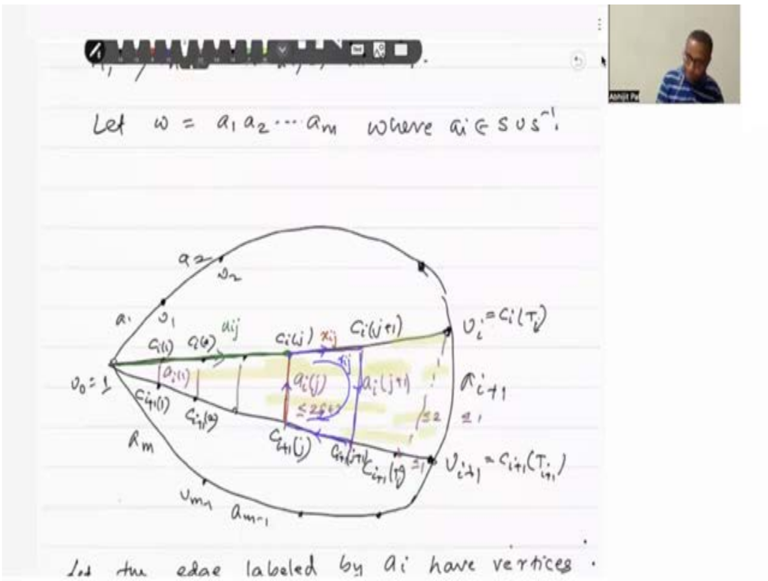

Now, let us define R as the set of words in G that have a length of at most 4δ + 6. Since the group G is finitely generated, it follows that R is a finite set. Furthermore, each of the elements

r_{ij} and r_i belongs to this set R. Now, we are nearly finished! What we need to do next is define u_{ij} as the word that represents the geodesic connecting the identity element 1 and the point c_{ij}. Thus, u_{ij} serves as a representation of this geodesic, illustrating the direct path from the identity to c_{ij} in the Cayley graph.

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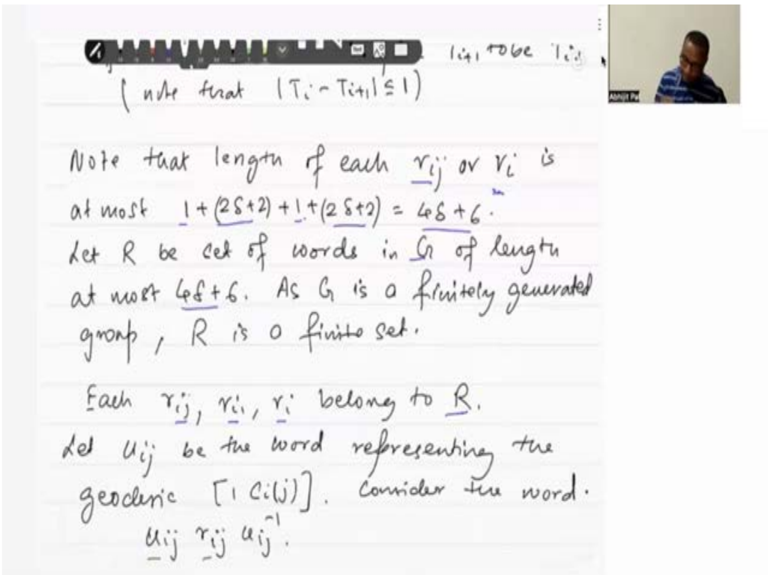

Now, let's consider the word $u_{ij}r_{ij}u_{ij}^{-1}$. We start by labeling this loop with u_{ij}, then we proceed to label it with r_{ij}, and finally, we return to the identity by labeling it with u_{ij}^{-1}.

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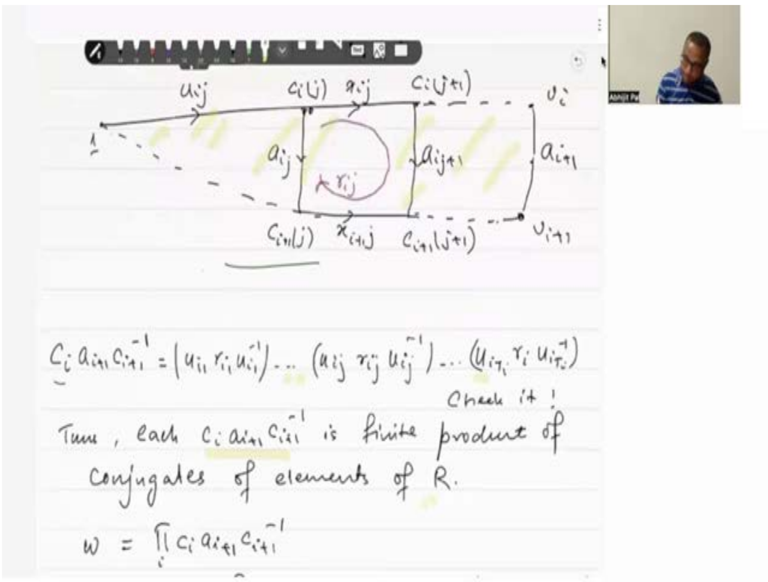

This sequence illustrates the journey: first, we travel along u_{ij}, then follow the loop r_{ij}, and finally, we conclude the path by returning back through u_{ij}^{-1}.

Now, we have this illustration in front of us. You can verify that the loop $c_i a_{i+1} c_i^{-1}$ is indeed the product of these smaller loops: $u_{i1} r_{i1} u_{i1}^{-1}, \dots, u_{ij} r_{ij} u_{ij}^{-1}$, and finally, $u_{i-1} r_i u_{i-1}^{-1}$. Importantly, each of these u terms represents a finite product of conjugates of elements from the set R, where r_{i1}, r_{ij}, and r_i all belong to this finite set R.

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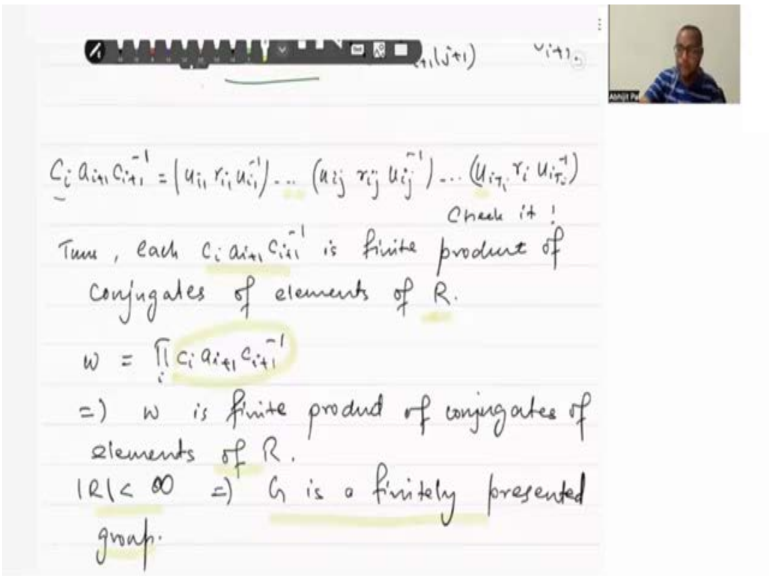

Thus, we see that w is indeed the product of the loops $c_i a_{i+1} c_{i+1}^{-1}$, and each of these components is a finite product of conjugates. Consequently, we can conclude that w is a finite product of conjugates of elements from the set R. Since R is a finite set, we have successfully demonstrated that the group G is a finitely presented group. With that, I will conclude this discussion.