An Introduction to Hyperbolic Geometry

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Lecture – 35

Understanding Quasi-Isometry: The Svarc-Milnor Lemma and Its Implications

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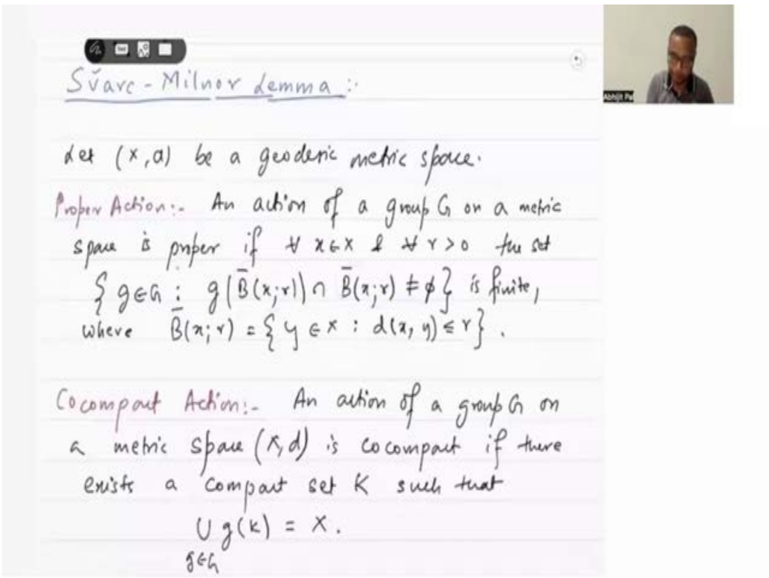

Hello! In the previous lecture, we discussed quasi-isometries and the Cayley graph of finitely generated groups. Today, we will move forward and examine an important result known as the Svarc-Milnor Lemma. This lemma states the following: if a group acts properly and co-compactly by isometries on a proper geodesic metric space, then that group is finitely generated, and furthermore, the Cayley graph of the group is quasi-isometric to the geodesic metric space.

Let's begin by defining what we mean by a proper action. Consider a geodesic metric space X. The action of a group G on X is said to be proper if, for all points $x \in X$ and for any positive radius r, the set of group elements $g \in G$ that translate the closed ball of radius r around x in such a way that the translated ball intersects the original ball is finite. In simpler terms, this means that the set of group translations that overlap with the ball around x is limited to a finite number of elements.

If this condition holds, we say that the action of the group on the metric space X is proper.

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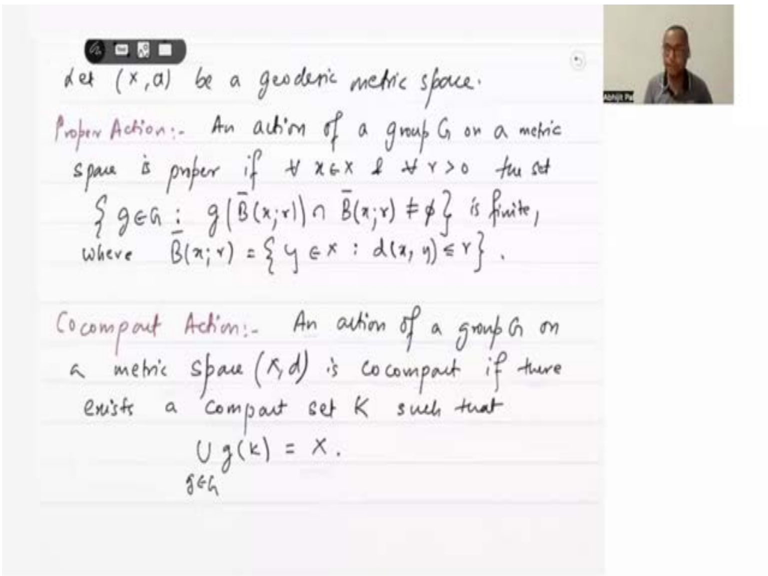

Now, what is the definition of a co-compact action? We've encountered this concept earlier. An action of a group G on a metric space X is called co-compact if there exists a compact set $K \subset X$ such that the union of all its translates under the group action covers the entire space X. In other words, the union of the sets gK, where $g \in G$, is equal to X.

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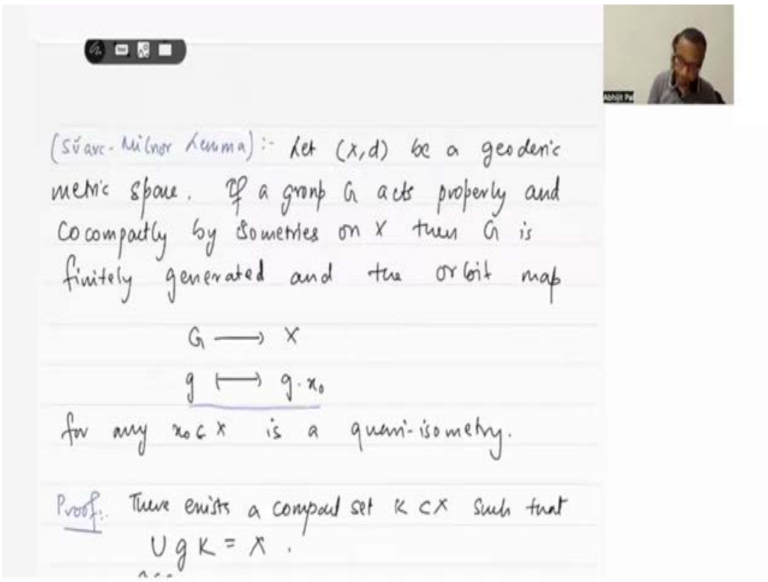

Now, let me state the Svarc-Milnor Lemma clearly. Let X be a geodesic space, and suppose a group G acts properly and co-compactly by isometries on X. In that case, G is a finitely generated group, and the orbit map $G \to X$, which maps an element $g \in G$ to $g \cdot x_0$ (where x_0

is a fixed point in X), is a quasi-isometry.

Furthermore, this map can be naturally extended to the Cayley graph of G, and once you establish that this map is a quasi-isometry, the Cayley graph will also be quasi-isometric to the geodesic space X.

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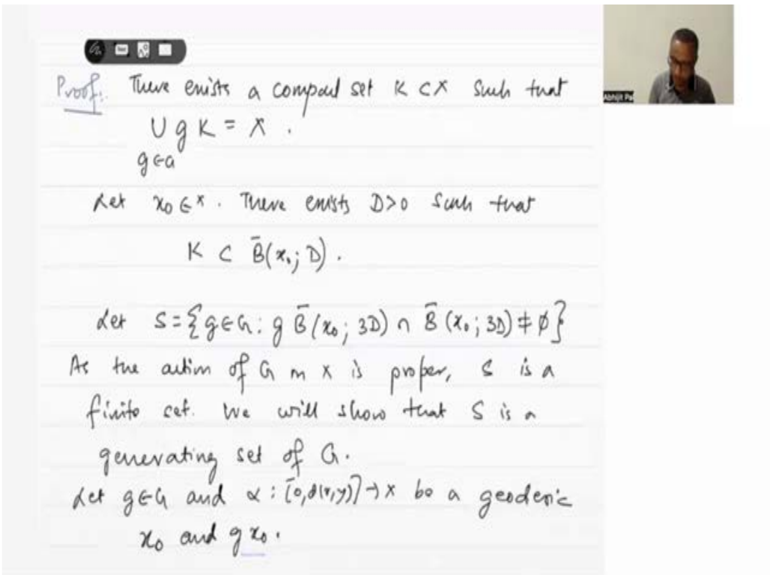

Let's begin with the proof. Since the action is co-compact, we know that there exists a compact set K such that the union of the translates of this set K covers the entire metric space X. Now, consider a point $x_0 \in X$. There exists a positive constant D such that the compact set K is contained within a closed ball centered at x_0 with radius D. This is due to the compactness of K, which implies that it must be bounded.

Now, define the set S as consisting of all elements of the group G such that the translate of the closed ball of radius 3D around x_0 intersects with the closed ball of radius 3D around x_0. In other words, $S = \{g \in G |\ \mathrm{g} \cdot B(x_0, 3D) \cap B(x_0, 3D) \neq \emptyset\}$

Since the action is proper, this set S will be finite. We will now show that this finite set S is, in fact, a generating set for the group G. And because S is finite, it follows that G is finitely generated.

Next, take any element $g \in G$, and let α be a geodesic in X from x_0 to $g \cdot x_0$. So, let α be a geodesic from x_0 to $g \cdot x_0$. In this context, d(x_0, y) should be understood as $d(x_0, g \cdot x_0)$, where y represents $g \cdot x_0$.

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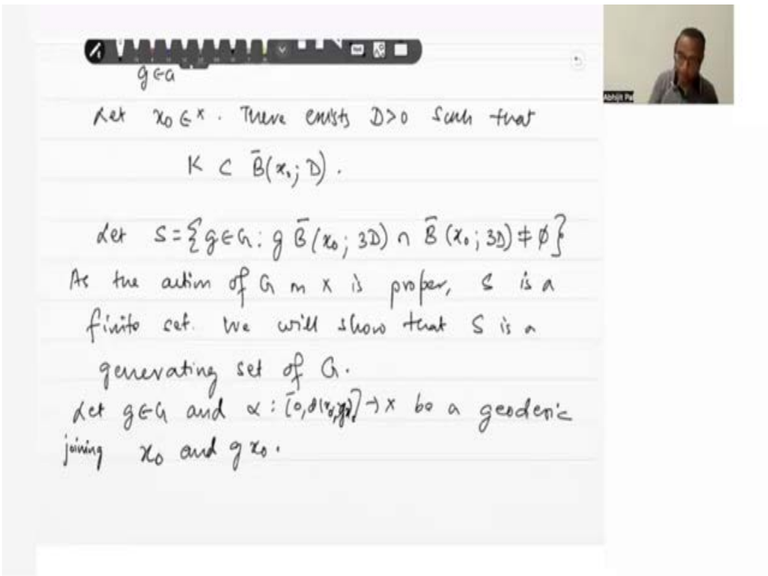

Therefore, α is defined as a map from the closed interval $[0, d(x_0, g \cdot x_0)]$ into the space X. Essentially, α is a geodesic path that connects the points x_0 and $g \cdot x_0$ in the metric space.

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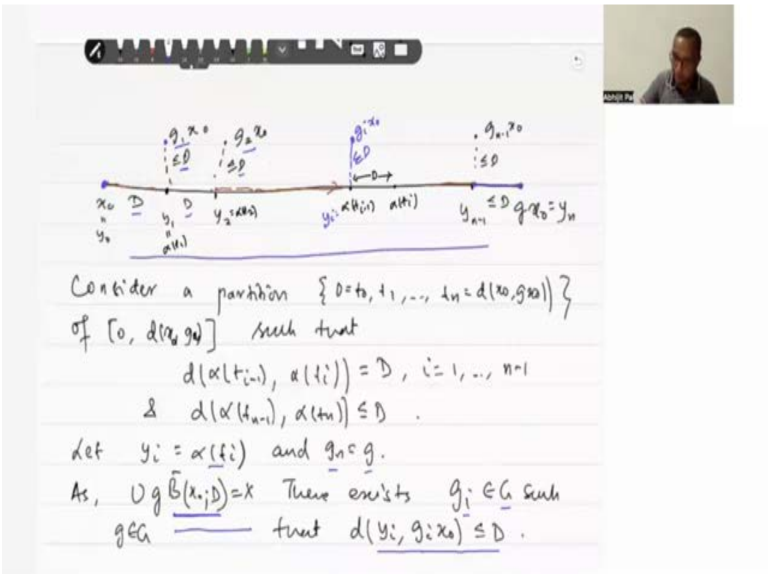

Now, let's look at the geodesic α. We proceed by partitioning the closed interval from 0 to $d(x_0, g \cdot x_0)$. Consider a partition $t_0, t_1, \dots, t_n$ such that the distance between $\alpha(t_{i-1})$ and $\alpha(t_i)$ equals D for i = 1 to n-1, and the distance between $\alpha(t_{n-1})$ and $\alpha(t_n)$ is less than or equal to D. This partitioning is always possible.

Now, imagine the geodesic α, where we mark points $\alpha(t_1), \alpha(t_2), \ldots, \alpha(t_{n-1})$ such that each segment from $\alpha(t_{i-1})$ to $\alpha(t_i)$ has a length exactly equal to D for i = 1 to n-1. In the last segment, the distance will be less than or equal to D.

Next, let $y_i = \alpha(t_i)$, and let $g_n = g$. Now, since X is the union of the translates of balls around x_0 with radius D, this holds because we know that $X = \cup\, G \cdot K$, where K is contained within the closed ball around x_0 with radius D. Thus, X is the union of translates of this closed ball around x_0.

From this, we conclude that for each point y_i, there exists an element $g_i \in G$ such that the distance between y_i and $g_i \cdot x_0$ is less than or equal to D. This follows directly from the structure of the space.

Now, by applying the triangle inequality, we observe the following: the distance between x_0 and $g_1 \cdot x_0$ is less than or equal to 2D, since it is bounded by D + D. Similarly, the distance between $g_1 \cdot x_0$ and $g_2 \cdot x_0$ will be less than or equal to 3D, as it sums up to D + D + D. Thus, the distances between the points follow this pattern, which provides us with the necessary bounds.

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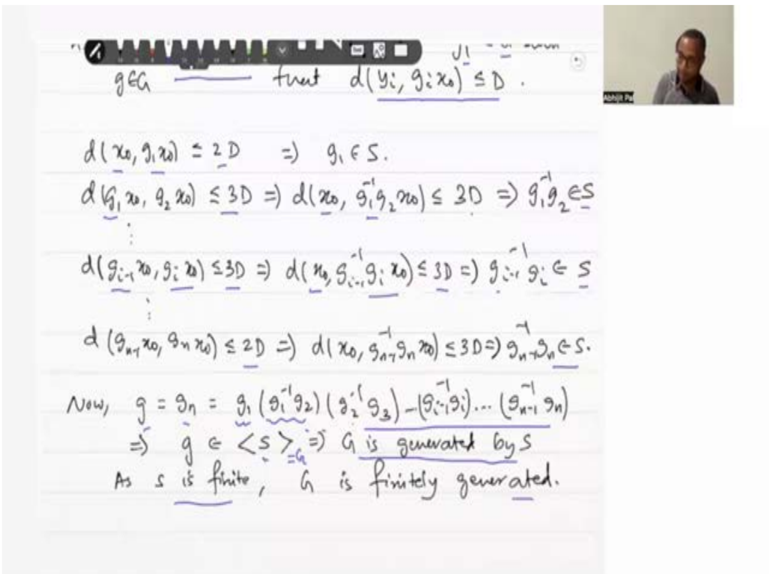

Now, let's examine the distances between the points. The distance between $g_1 \cdot x_0$ and $g_2 \cdot x_0$ is less than or equal to 3D. Since this group is acting by isometries, the distance between x_0 and $g_1^{-1} g_2 \cdot x_0$ will also be less than or equal to 3D. This implies that the group element $g_1^{-1} g_2$

belongs to the set S.

Similarly, for any i less than n, the distance between $g_{i-1} \cdot x_0$ and $g_i \cdot x_0$ is also less than or equal to 3D. Thus, the distance between x_0 and $g_{i-1}^{-1} g_i \cdot x_0$ is less than or equal to 3D, which implies that the group element $g_{i-1}^{-1} g_i$ also belongs to S.

When i = n, the distance between $g_{n-1} \cdot x_0$ and $g_n \cdot x_0$ will be less than or equal to 2D, which means the group element g_{n-1}^{-1} g belongs to S.

Now, we began with a group element g = g_n, and this g_n can be expressed as:

$$g_n = g_1 \cdot (g_1^{-1} g_2) \cdot (g_2^{-1} g_3) \cdots (g_{n-1}^{-1} g_n)$$

Each of the elements $g_1^{-1} g_2, g_2^{-1} g_3, \ldots$, all belong to the set S, and since g_1 also belongs to S, we can conclude that the group element g belongs to the group generated by S.

Therefore, the group G is generated by the finite set S, meaning that G is finitely generated.

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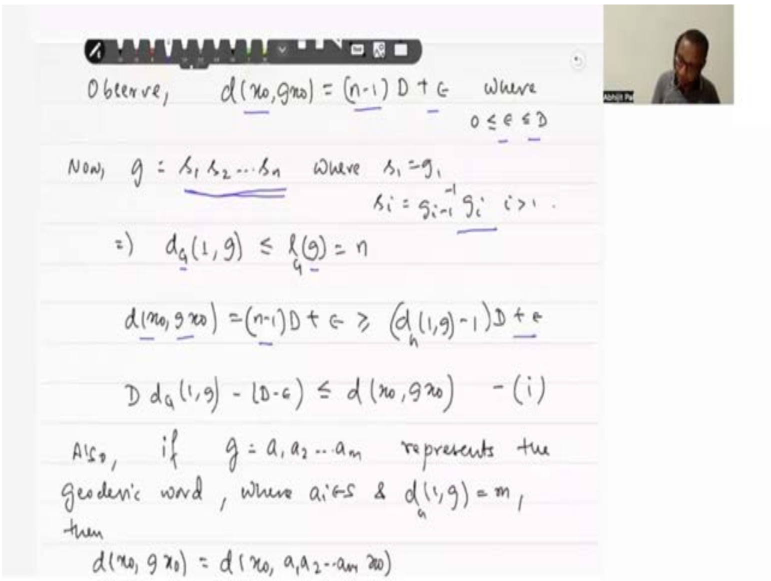

Now, let us proceed to prove that the group G is quasi-isometric to the space X. First, observe that the distance between x_0 and $g \cdot x_0$ can be expressed as $(n-1) \cdot D + \epsilon$, where ϵ is a small positive value such that $0 \leq \epsilon \leq D$. This result follows directly from the partition we previously established.

Recall that we expressed the group element g as $g = s_1, s_2, \ldots, s_n$, where $s_1 = g_1$ and $s_i =$

$g_{i-1}^{-1} g_i$ for all i. The distance between the identity element and g in the group G is less than or equal to the length of the element g, and this length is exactly equal to n, as we derived from the partition.

Thus, we have that the distance between x0 and $g \cdot x_0 is (n-1) \cdot D + \epsilon$, and n is greater than or equal to the distance between the identity element and g in the group G. Moreover, this distance is greater than or equal to the distance between the identity and g, minus D, plus ϵ.

This setup directly implies that the group G and the space X are quasi-isometric, as the distances in the group and the metric space are comparable within constant factors, as we have shown.

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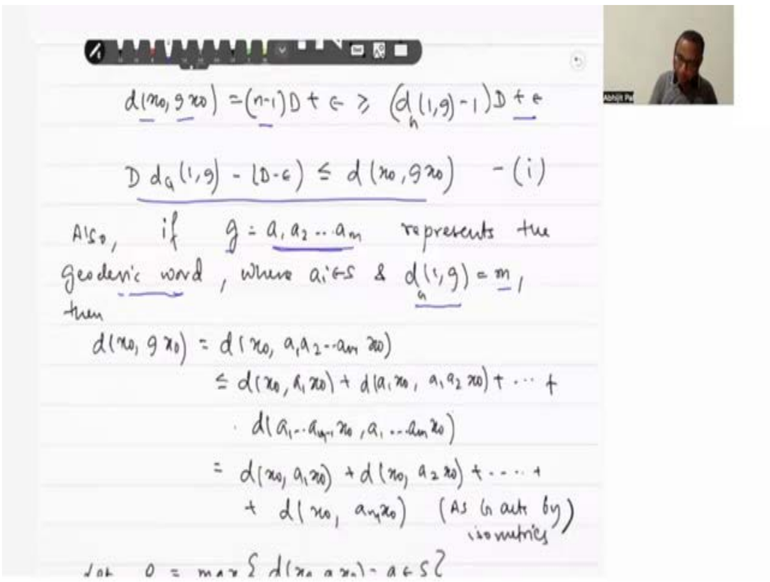

Thus, we have established the inequality. Now, if $g = a_1, a_2, \ldots, a_m$, where this sequence represents a geodesic word, it means the length of g is equal to the distance between the identity element 1 and g in the group G. Hence, the distance between 1 and g is precisely m, since g is represented by this geodesic word.

Next, consider the distance between x0 and $g \cdot x_0$, which equals the distance between x0 and the point $a_1, a_2, \ldots, a_m \cdot x_0$. By applying the triangle inequality, we get:

$$\begin{aligned}
&\text{distance}(x_0, g \cdot x_0) \\
&\qquad \le \text{distance}(x_0, a_1 \cdot x_0) + \text{distance}(a_1 \cdot x_0, a_1 a_2 \cdot x_0) + \cdots \\
&\qquad + \text{distance}(a_1 a_{m-1} \cdot x_0, a_1 a_m \cdot x_0)
\end{aligned}$$

Now, notice that the distance between $a_1 \cdot x_0$ and $a_1 a_2 \cdot x_0$ is exactly the same as the distance between x_0 and $a_2 \cdot x_0$, because the group G is acting by isometries. Thus, this pattern holds throughout, and ultimately, the distance between $a_1 a_{m-1} \cdot x_0$ and $a_1 a_m \cdot x_0$ is equal to the distance between x_0 and $a_m \cdot x_0$.

Therefore, the total distance between x_0 and $g \cdot x_0$ is bounded by the sum of these distances, affirming that the group G, acting by isometries, preserves the metric properties of the space X.

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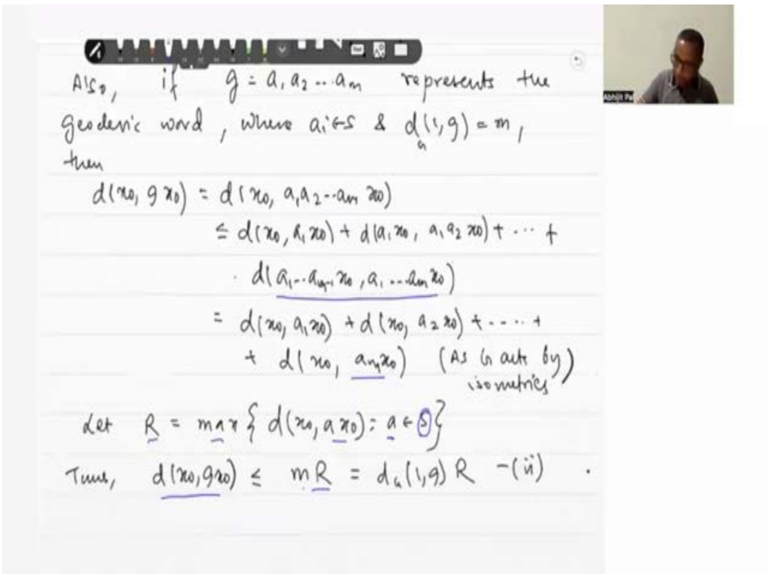

Now, let's define R to be the maximum distance between x_0 and $a \cdot x_0$, where a belongs to the finite set S. Since S is finite, this maximum R exists. Therefore, the distance between x_0 and $g \cdot x_0$ can be bounded as follows:

$$\text{distance}(x_0, g \cdot x_0) \leq m \times R$$

Here, m represents the distance between the identity element 1 and g in the group G. So, this bound provides a relationship between the action of g on the space and the length of g in the group, with R acting as a scaling factor due to the finiteness of S.

So, what have we proven? Let's take two elements g and g' that belong to the group G. We can express g as $g = g'^{-1} g''$. To clarify, let’s denote the elements in our group G as g' and g'', so we have $g = g'^{-1} g''$.

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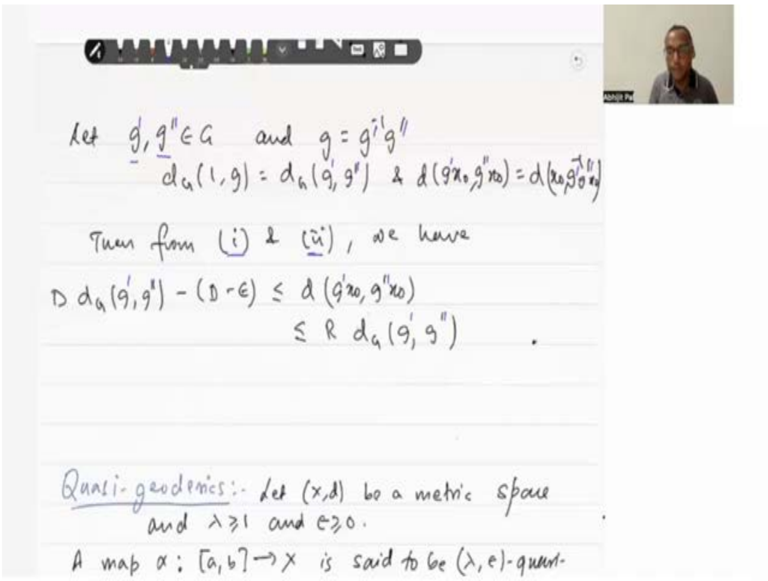

The distance between the identity element 1 and g in the group is equal to the distance between g' and g''. Furthermore, the distance between g' and g'' when applied to the point x_0 can be expressed as the distance between x_0 and $g'^{-1}g'' \cdot x_0$.

From our previous equations, we derive the following inequality:

$$D \cdot \text{distance}(g', g'') - D - \epsilon \leq \text{distance}(g' \cdot x_0, g'' \cdot x_0) \leq \text{distance}(g', g'')$$

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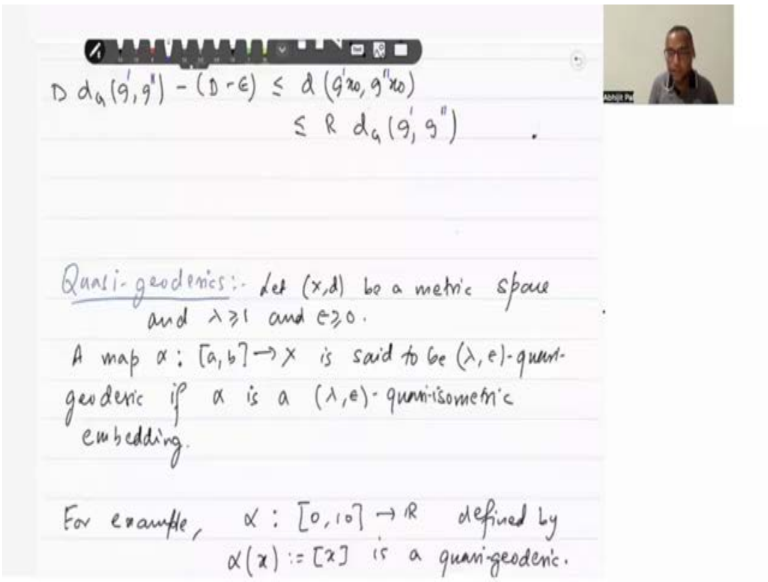

This inequality clearly indicates that the map from G to x, defined by $g \mapsto g \cdot x_0$, is indeed a quasi-isometry.

I will conclude our discussion here for today. In our next class, we will delve into the concept of quasi-geodesics and prove an important property related to them in a hyperbolic metric space. Specifically, we will explore the relationship between a geodesic and a quasi-geodesic that connects the endpoints of the quasi-geodesic. It turns out that these two paths lie within a uniformly bounded neighborhood of each other. This fascinating property is referred to as the stability of quasi-geodesics.