An Introduction to Hyperbolic Geometry

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Lecture – 34

Cayley Graphs of Finitely Generated Groups and Quasi-Isometric Relations

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Hello everyone! In this lecture, we will begin by exploring the Cayley graph of a finitely generated group, and then we will delve into the concept of quasi-isometries. For any finitely generated group, you can construct a Cayley graph, and if you change the generating set, you will obtain two different Cayley graphs that are quasi-isometric to each other. So, let's start with the definition of the Cayley graph.

To begin, let's consider a finitely generated group. Let G be a finitely generated group with a finite generating set S. This means that G is generated by the finite set S. We will define a graph, denoted as γ(G, S), which we will refer to as the Cayley graph. The vertex set of this graph consists of the entire group G. To properly define a graph, we need a vertex set, and in this case, the vertex set $V_{\gamma(G,S)}$ is simply the group G.

Next, we need to define the edge set. Let's take two elements from this vertex set. Let g, g' belong to this vertex set. The vertices g and g' will have an edge between them if $g^{-1}g'$ belongs to S, or equivalently, if $g^{-1}g'$ is in S. Additionally, we impose another condition: if there exists

some a belonging to S such that a^2 equals the identity element, denoted by 1, then there will be two edges between 1 and a, one labeled a and the other labeled a^{-1}.

For instance, if $a^2 = 1$, then it follows that $a = a^{-1}$. In this case, we will have an edge from 1 to a labeled a, and there will be another directed edge from a back to 1 labeled a^{-1}. This is how we construct the Cayley graph for a finitely generated group.

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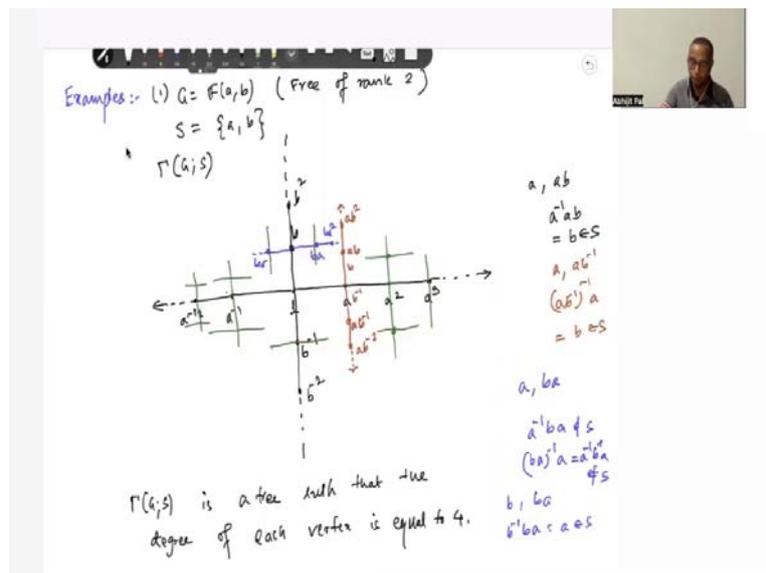
Now, let's move on to some examples. We will start with the free group of rank 2, denoted as G. The generating set for this group consists of the elements a and b. Our goal is to draw the corresponding Cayley graph.

First, we begin with the identity element, which we'll denote as 1. When we take the generator a, we note that $1^{-1} a$ belongs to the set S. Therefore, we can draw an edge between the vertices 1 and a. Next, let's consider a and a^2. When we look at $a^{-1} a^2$, we find that it simplifies to a, which also belongs to S. Consequently, there will be an edge connecting a and a^2.

It's important to note that if we take 1 and a^2, the expression $1^{-1} a^2$ simplifies to a^2 and does not belong to S. Thus, we will not have an edge between 1 and a^2. Now, considering a^{-1}, we can draw an edge between a^{-1} and 1. Continuing this process, we can also establish an edge between a^2 and a^3. This will yield a line corresponding to the cyclic subgroup generated by a.

Next, let's introduce the generator b. From 1 to b, we can again draw an edge. The same holds for b^2; there will be an edge connecting 1 and b^{-1}, and similarly between b and b^{-2}. You can

continue this process indefinitely.

Now, let's consider the word ab. Taking the other generator a, we can examine a^{-1} a b. This expression simplifies to b, which belongs to S. Therefore, there will be an edge between a and ab. We can represent this edge with the label b.

Now, if we consider the elements a and a^{-1} b, or rather ab^{-1}, we need to check the relationship between a and ab^{-1}. If we take ab^{-1} and find its inverse, we can analyze the corresponding edges.

So, when we consider the word a b^{-1}, we see that it simplifies to b, which belongs to the set S. As a result, there will be an edge connecting a and a b^{-1}. We can label this edge with b^{-1}. Consequently, we also have an edge leading to the vertex a b^2. This process can be continued in the same manner.

Next, let's examine the word b a. If we take the elements a and b a, we find that a^{-1} b a does not belong to S. Similarly, if we consider b a^{-1} a, we find that this also simplifies to a^{-1} b^{-1} a and does not belong to S. Therefore, we conclude that there is no edge between a and b a.

However, if we look at the two words b and b a, we find that b^{-1} b a simplifies to a, which does belong to S. Consequently, there will be an edge between b and b a. This clearly illustrates that the group G, which is a free group generated by the elements a and b, is non-commutative; in other words, b a is not equal to a b.

Now, similarly in the opposite direction, we will obtain b a^{-1}, resulting in an edge connecting b^2. It is important to note that this blue horizontal line will not intersect with the red line. Thus, for each of these vertices, we will either see a vertical line or a horizontal line. For every vertex, there are two kinds of lines: one vertical and one horizontal.

These lines will only intersect at a single point. So, what is the Cayley graph? The Cayley graph $\Gamma(G, S)$ is structured such that the degree of each vertex is equal to 4. In fact, it forms an infinite tree, specifically an infinite regular tree where each vertex has a degree of 4.

Now, let's consider another example. We will take the group G to be $Z \times Z$, which can be presented in the following way. This group is generated by two elements, a and b, with the relation a b a^{-1} b^{-1} = 1. This gives us a presentation for the group $Z \times Z$. Corresponding to the cyclic group generated by a, we can visualize this as a horizontal line, and for the generator b, we can represent it as a vertical line.

Now, considering the elements ab and a, we will find that there is an edge connecting them. In fact, starting from a, you will have a vertical line leading to a vertex, which we can label as ab, followed by ab^2. It's important to note that in $Z \times Z$, we have ab = ba. Therefore, there will be an edge between b and ba. As a result, we will also have a horizontal line connecting these two points.

So, the Cayley graph of this group can be visualized as a lattice in R^2. This lattice effectively represents the Cayley graph of the group.

For the third part, I will leave this as an exercise for you: consider a group generated by a and b where the relation is $a^2 = b^3$. One can prove that this group is isometric to Z. In this case, the Cayley graph of Z is simply a horizontal line. However, the Cayley graph of this specific group will differ. Despite this difference, there exists a concept known as quasi-isometry. It can be shown that the Cayley graphs of these groups are quasi-isometric to each other.

With that, we are now in a position to define quasi-isometry, so let's proceed with that explanation.
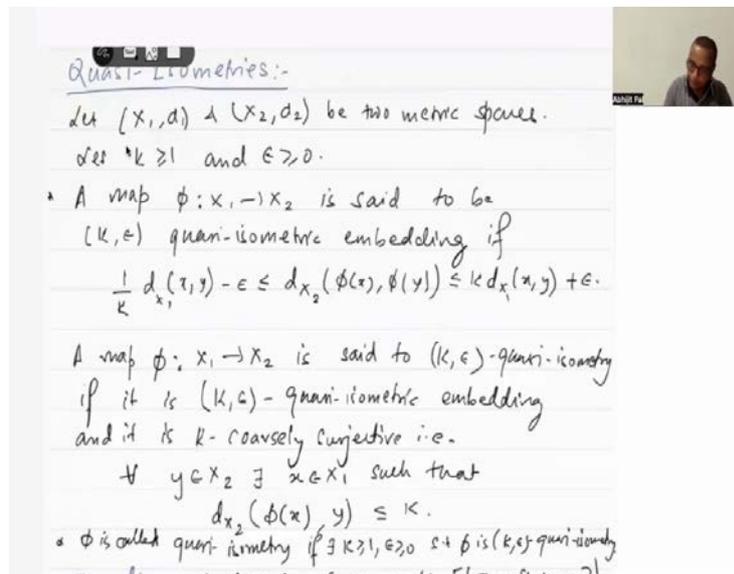
Let's begin with two metric spaces, denoted as X_1 and X_2, equipped with metrics d_1 and d_2 respectively. We will also introduce two parameters: k, which is greater than or equal to 1, and ϵ, which is greater than or equal to 0.

A map φ from X_1 to X_2 is said to be a k ϵ-quasi-isometry embedding if the following inequality

holds:

$$\frac{1}{k} \cdot d_1(x, y) - \epsilon \leq d_2\big(\phi(x), \phi(y)\big) \leq k \cdot d_1(x, y) + \epsilon.$$

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Here, it's important to note that if we set $\epsilon = 0$ and $k = 1$, then φ behaves as an isometric embedding. Thus, a k ϵ-quasi-isometry embedding serves as a generalization of the isometric embedding concept.

Furthermore, we refer to a map φ from X_1 to X_2 as a k ϵ-quasi-isometry if it satisfies two conditions:

1. The map φ is a k ϵ-quasi-isometry embedding, and

2. It is k-coarsely surjective.

This means that for any element y in the codomain space X_2, there exists a point x in the domain space X_1 such that the distance between φ(x) and y is less than or equal to k.

If we were to consider the case where k = 0, this would imply that the map φ is surjective. However, since we are stipulating that k must be greater than or equal to 1, this indicates that φ is coarsely surjective.

In conclusion, we define a map φ as a quasi-isometry if there exist numbers k (where $k \geq 1$)

and ϵ (where $\epsilon \geq 0$) such that the map φ qualifies as a k ϵ-quasi-isometry.

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Let's consider an example involving the Cayley graph of the group $Z \times Z$. In this case, we can take the generating set to be (0, 1) and (1, 0). We can also represent these generators as a and b, with the relation given by $ab^{-1}b^{-1} = 1$.

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The Cayley graph for this configuration will manifest as a square lattice in R^2. This lattice structure visually illustrates the relationships defined by the generators and their connections

within the group, highlighting the nature of $Z \times Z$ as it maps onto the Cartesian plane.

Now, let's discuss the metric defined on this graph. For two points (p₁, q₁) and (p₂, q₂) within the graph Γ, the distance is calculated as the absolute value of the difference in the p-coordinates, added to the absolute value of the difference in the q-coordinates. In mathematical terms, this can be expressed as:

$$d_\Gamma\big((p_1, q_1), (p_2, q_2)\big) = |p_1 - p_2| + |q_1 - q_2|$$

To illustrate this, consider two points on the graph. The distance metric can be understood as the sum of two segments: first, you move horizontally from one point to the other, taking the length of this horizontal segment; then, you move vertically, measuring the length of the vertical segment. By adding these two lengths together, you arrive at the total distance between the two points.

This metric defines the structure of the graph Γ, which naturally embeds into R^2. In this two-dimensional space, the standard metric is the Euclidean metric, which measures distance in a familiar geometric sense. Thus, we can see that the graph's metric aligns with the principles of the Euclidean metric in R^2.

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One can demonstrate that the distance in R^2 is less than or equal to the distance in the graph Γ, which in turn is less than or equal to twice the distance in R². Moreover, if we consider any

point (x, y) in R^2, there exists a point on Γ such that the distance from (x, y) to that point is less than or equal to $\frac{1}{2}$. This is clearly illustrated in the accompanying diagram, which shows that the distance from (x, y) to Γ is indeed less than or equal to $\frac{1}{2}$. Therefore, we conclude that the embedding of Γ in R^2 qualifies as a quasi-isometry.

Now, let's consider a finitely generated group G with two finite generating sets, S_1 and S_2. This means that G is generated by S_1 and also by S_2, with both sets being finite. Consequently, we have an identity map from the group generated by S_1 to the group generated by S_2. In fact, this identity map is a quasi-isometry.

Furthermore, this identity map induces a quasi-isometry on the Cayley graph corresponding to these groups. Let's define S_1 as the set $\{a_1, a_2, \ldots, a_n\}$ and S_2 as the set $\{b_1, b_2, \ldots, b_m\}$. Denote the groups generated by these sets as G_i, where $i = 1, 2$. It is important to note that both G_1 and G_2 are indeed equal to G.

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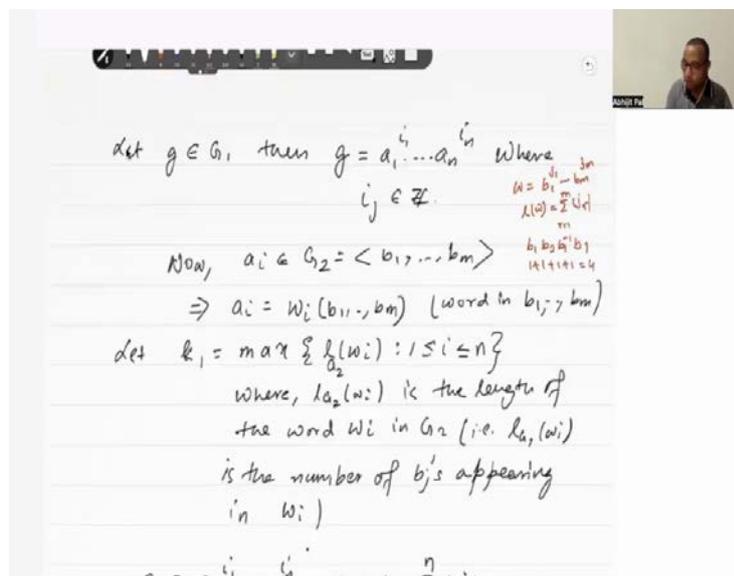
Now, let us consider an element g that belongs to G_1. This element g can be expressed as $g = a_1^{i_1} \cdots a_n^{i_n}$, illustrating that the group G_1 is generated by the elements $a_1, a_2, \ldots, a_n$. Therefore, any element within G_1 can indeed be represented as a product of these generators. This notation captures the essence of the structure of G_1.

It is also important to note that each a_i belongs to G_2 as well, confirming that G_1 and G_2 are, in

fact, the same group. In this case, G_2 is generated by the elements $b_1, b_2, \ldots, b_m$. Consequently, since each a_i is an element of G_2, we can conclude that each a_i can be expressed as a word composed of the generators $b_1, b_2, \ldots, b_m$.

Next, let's define the length of a word w_i in G_2. Here, the length of a word w, such as $w = b_1^{j_1} b_2^{j_2} \cdots b_m^{j_m}$, is determined by the sum of the absolute values of the exponents j_r. For instance, if we consider the word b_1 b_2 b_1^{-1} b_1, this particular word may not be in its reduced form, but the length is calculated as $1 + 1 + 1 + 1$, yielding a total length of 4.

To facilitate further analysis, let us define k_1 as the maximum length of the word w_i in G_2. This length represents the number of b_j elements that appear in w_i.

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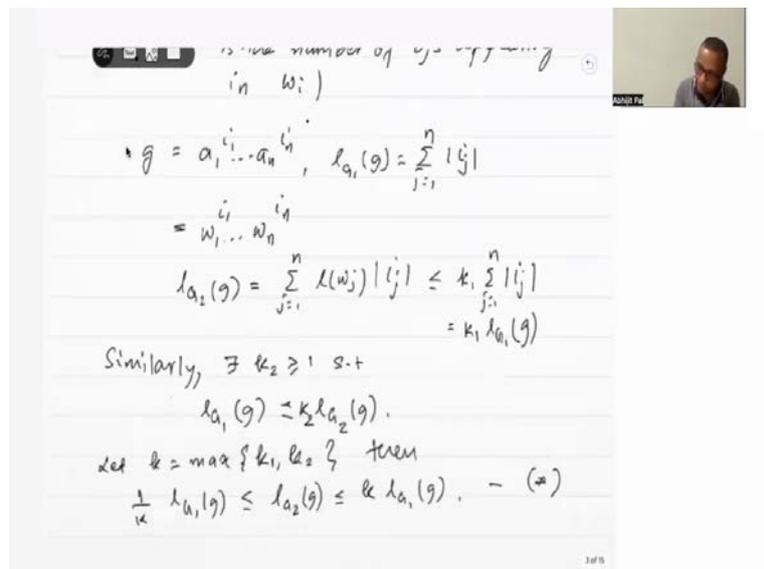
Now, let us examine the element g again, expressed as $g = a_1^{i_1} \cdots a_n^{i_n}$ within G_1. The length of this element g in G_1 can be defined as the sum $\sum_{j=1}^{n} |i_j|$. Each of the elements a_i corresponds to a word w_i. Thus, we establish the equality that relates these lengths.

When we analyze the length of the word g in G_2, we find that it can be represented as $\sum_{j=1}^{n} \text{length}(w_j) \cdot |i_j|$. Given that the length of w_j is constrained by our earlier definition, specifically, $\text{length}(w_j) \le k_1$, we can infer that the length of g in G_2 satisfies the inequality:

$$\text{length}(g \text{ in } G_2) \le k_1 \cdot \sum_{j=1}^{n} |i_j|$$

This means that the length of g in G_2 is less than or equal to k_1 times the length of g in G_1.

In a similar fashion, we can assert the existence of a constant k_2 such that the length of g in G_1 is less than or equal to k_2 times the length of g in G_2:

$$\text{length}(g \text{ in } G_1) \leq k_2 \cdot \text{length}(g \text{ in } G_2)$$

Now, let us define k to be the maximum of k_1 and k_2. Consequently, we arrive at the following conclusion:

$$\frac{1}{k} \cdot \text{length}(g \text{ in } G_1) \leq \text{length}(g \text{ in } G_2) \leq k \cdot \text{length}(g \text{ in } G_1).$$

This establishes a quasi-isometric relationship between the lengths of g in G_1 and G_2.

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Now, let us consider two elements, g and g', within the group G. To define the distance between g and g' in the group G_i where i = 1, we need to look at the infimum of the lengths of all possible words w that represent the element $g^{-1}g'$. It is important to note that there can be multiple words that represent $g^{-1}g'$, and our goal is to find the shortest among them.

From this definition, we can derive an important relationship. Specifically, we find that:

$$\frac{1}{k} \cdot \text{distance}(g, g') \text{ in } G_1 \leq \text{distance}(g, g') \text{ in } G_2 \leq k_1 \cdot \text{distance}(g, g') \text{ in } G_1$$

This establishes that, with respect to this metric, the groups G_1 and G_2 are quasi-isometric to

each other.

Additionally, it is crucial to point out that this metric is referred to as the word metric. Therefore, we conclude that, under the word metric, the groups G_1 and G_2 are indeed quasi-isometric to one another. Thus, we have successfully proved this theorem.

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Let's consider a finitely generated group with two distinct finite generating sets, S_1 and S_2. In this context, the group generated by S_1 will be quasi-isometric to the group generated by S_2 when we use the word metric.

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Now, if we examine the Cayley graph associated with this group, we find that the vertex set of the Cayley graph is indeed the group G itself. Furthermore, under the word metric, the distance between the vertices in the Cayley graph corresponds directly to the word metric.

This means that, with respect to the word metric, these groups are quasi-isometric to each other. Moreover, the vertex set is nicely embedded in the Cayley graph; in fact, it is quasi-isometrically embedded. Consequently, the Cayley graphs themselves will also be quasi-isometric to one another, and verifying this property is quite straightforward. I encourage you to take this as an exercise to reinforce your understanding.

I will pause here, and in the next class, we will delve into the Svarc-Milnor Lemma. This lemma asserts that if a group acts properly and co-compactly on a geodesic metric space X, and we also assume that this group is finitely generated, then the Cayley graph of that group will be quasi-isometric to the geodesic metric space X. So, I will stop it.