An Introduction to Hyperbolic Geometry

Prof: Abhijit Pal

Department of Mathematics & Statistics

Indian Institute of Technology - Kanpur

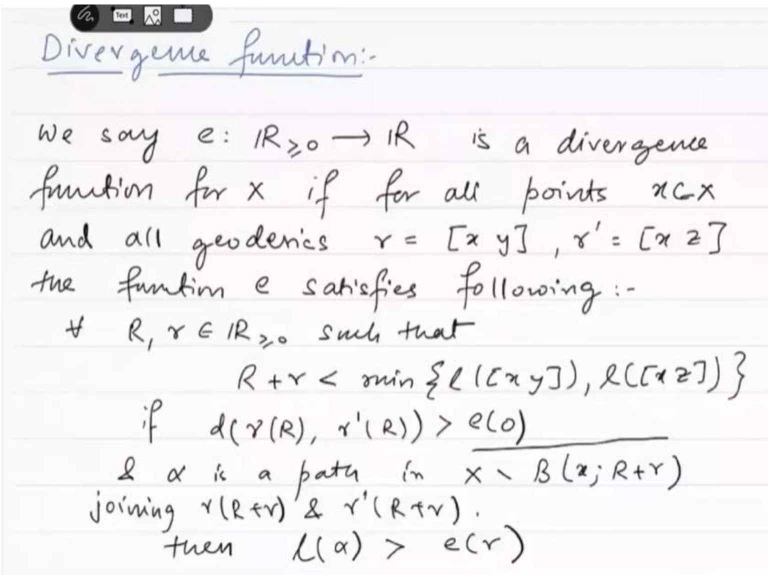
Module No. 07

Lecture No. 33

Characterization of Non-linear Divergence Functions in Metric Spaces

Hello! In our previous lecture, we introduced the concept of a divergence function and established that, within a hyperbolic metric space, there exists a divergence function that exhibits exponential behavior. In this lecture, we will prove that if a metric space possesses a non-linear divergence function, then it necessarily qualifies as a hyperbolic metric space. To begin, let us revisit the definition of a divergence function.

(Refer Slide Time: 00:59)

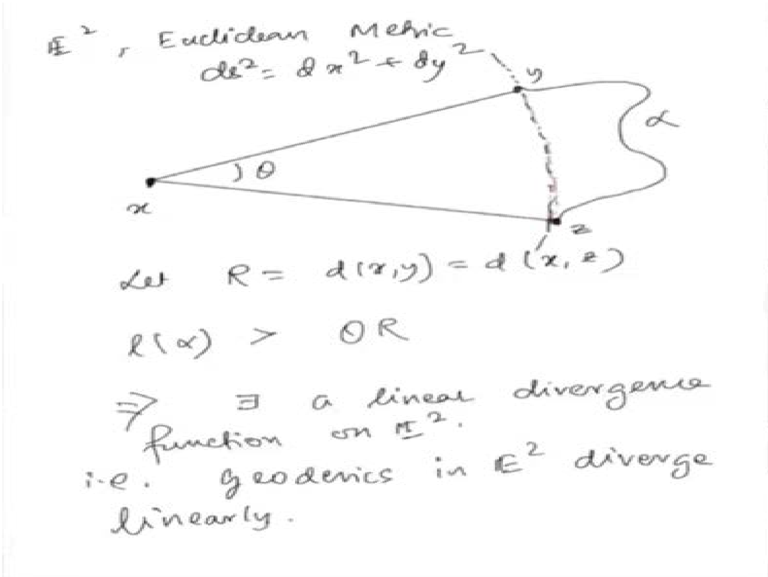


We define a function $e: R_{\geq 0} \to R$ as a divergence function for a geodesic metric space X if it satisfies certain conditions for all points x in the space and for all geodesics γ from x to y and γ' from x to z. Specifically, for all non-negative numbers R such that R + r is less than the minimum of the lengths of the geodesics xy and xz, we consider the scenario where the distance between γ R and γ' R is greater than e_0.

Now, let α be a path lying in the closure of the complement of the ball of radius R + r around x, where α connects γ R + r to γ' R + r. Under these conditions, the length of α is greater than e(R).

So, what occurs in Euclidean space? Let's explore that further.

(Refer Slide Time: 02:39)



Let us consider the Euclidean plane equipped with the Euclidean metric. We start with a point x in E^2 and draw two straight lines emanating from this point. We will label the intersection of one line with another point y and the other line with point z. Next, we define r to be the distance between x and y, ensuring that the distance from x to z is also equal to r. We will denote the angle between these two lines as θ.

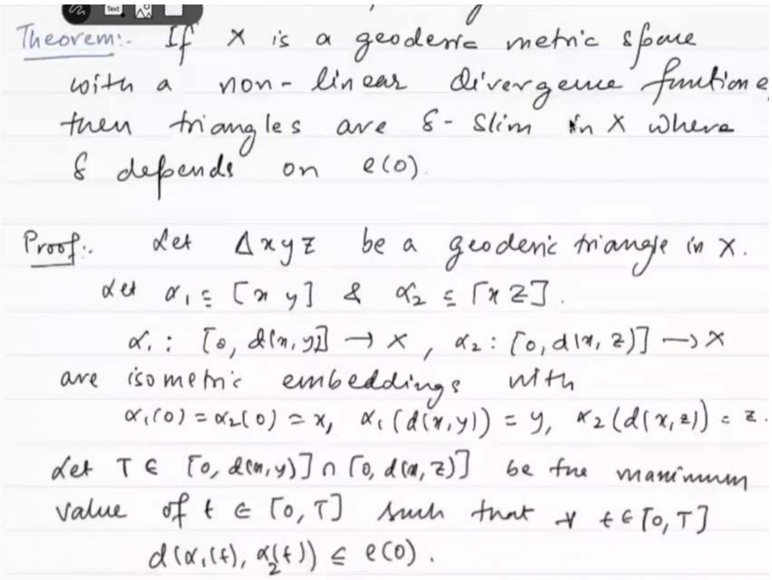
Now, if we consider a ball of radius R centered at point x and take any path α that lies outside this ball, it follows that the length of α is always greater than θ r. Here, θ r represents the length of the arc corresponding to the angle θ. This demonstrates that there exists a linear divergence function in E^2, indicating that geodesics in E^2 diverge linearly.

Now, we are poised to show that if a non-linear divergence function exists, then the corresponding geodesic metric space must indeed be a hyperbolic metric space.

Let us present a theorem: Let x be a geodesic metric space that possesses a non-linear divergence function e. We will prove that the triangles within this space are delta-slim, where this delta is determined solely by e(0).

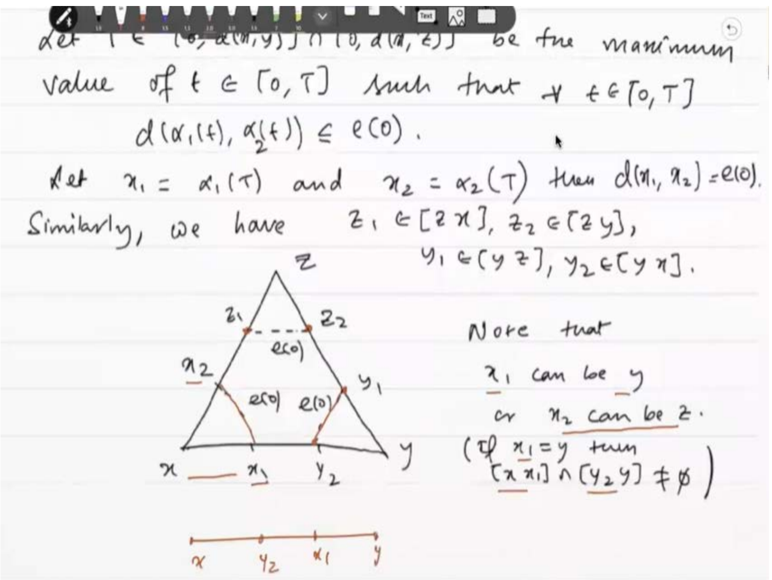
To illustrate, consider a triangle x, y, z within our geodesic metric space. We will designate the geodesic side x y as α_1 and the geodesic x z as α_2.

(Refer Slide Time: 05:53)**



If we express α1 and α2 as mappings, we can define α1 as a map from the closed interval $[0, d_{xy}]$ to x, and α2 as a map from the closed interval $[0, d_{xz}]$ to x. Importantly, both α1 and α2 are isometric embeddings. This means that for any s and t within the interval $[0, d_{xy}]$, the distance between α1(s) and α1(t) is given by the absolute value of the difference, |s - t|, and similarly for the mapping α2.

(Refer Slide Time: 07:23)



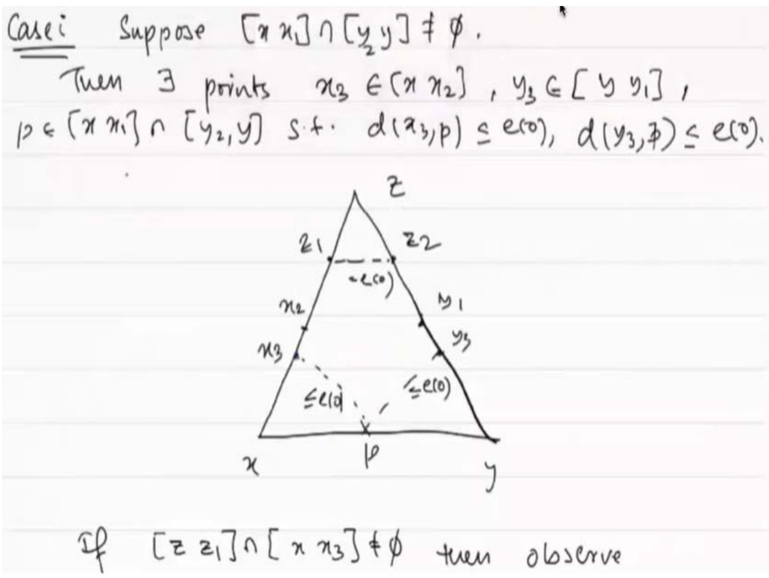
Now, let's introduce a point T in the intersection of these two intervals, chosen such that it is the maximum value for all t belonging to the interval [0, T] and satisfying the condition that the distance between $\alpha_1(t)$ and $\alpha_2(t)$ is less than or equal to e_0. This will yield a maximum T such that the distance between $\alpha_1(T)$ and $\alpha_2(T)$ is precisely equal to e_0.

To clarify, we define $x_1 = \alpha_1(T)$ and $x_2 = \alpha_2(T)$, resulting in the distance between x_1 and x_2 being exactly e_0. This applies to the sides x y and x z, and we can replicate this process for the remaining two sides of the triangle.

For the segment corresponding to x y, we have identified the points x_1 and x_2 such that the distance between them is equal to e_0. Similarly, for side y z, we will find points y_1 and y_2 where the distance is also equal to e_0. Finally, on the side x z, we will denote the points z_1 and z_2 such that the distance between them is exactly e_0.

It is important to note that there are various possibilities regarding these points; for example, x_1 could be the same as y, or x_2 could coincide with z. If we consider the case where x_1 = y, then the intersection of the geodesics at x_1 and y will be non-empty. In such a scenario, the positioning of x_1 would look something like this in the diagram.

(Refer Slide Time: 10:21)



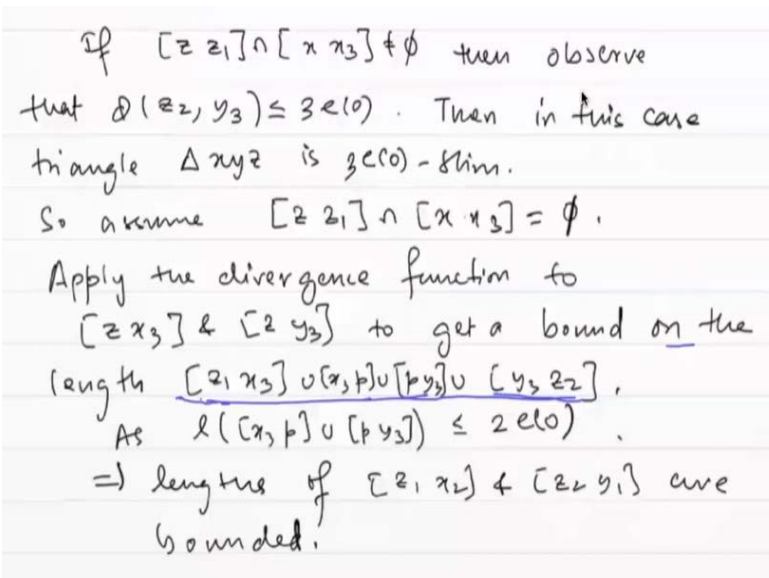
Even though this proof will hold, let’s first examine the initial case. We will assume that the geodesics x x_1 and y y_2 intersect with one another. In the diagram, we can visualize this scenario where x_1 is positioned here, leading us to identify the points x_3 on the geodesic x x_2 and y_3 on the geodesic y y_1. We denote the intersection of these two geodesics as point p.

The distances from x_3 to p and from y_3 to p are both less than or equal to e_0. Thus, point p is located at this intersection. Here, we can place point x_1 and, for instance, suppose point y_2 is situated nearby. We can conclude that there exists a point p and a point x_3 such that the distance between x_3 and p is less than or equal to e_0.

It is important to note that the distance between x_1 and x_2 is exactly e_0. If we take any point along the geodesic that is at a distance t from either x_1 or x_2, this distance will also remain less than or equal to e_0. Therefore, for point p, we can find point x_3 such that the distance between p and x_3 is less than or equal to e_0, while the distance from p to x will be equal to the distance from x to x_3.

Similarly, for the other side y z, we can identify point y_3 on the geodesic connecting y and y_1 such that the distance from p to y_3 is also less than or equal to e_0.

(Refer Slide Time: 13:20)



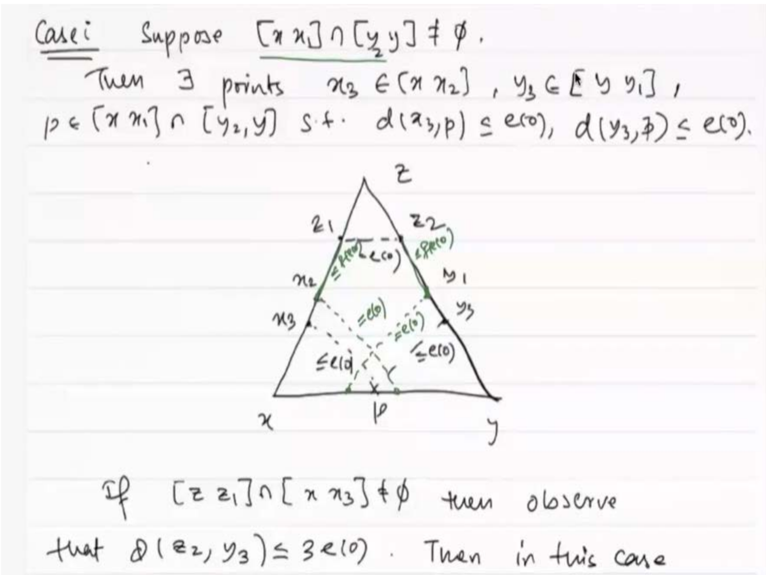
Now, let us consider the scenario where the intersection is not empty. This indicates that point z_1 will be situated somewhere within the diagram, likely in this area. Through a straightforward calculation employing the triangle inequality, we can establish that the distance between points z_2 and y_3 is less than or equal to 3 e_0. Consequently, in this particular case, we can conclude that this triangle is 3 e_0-slim.

We have successfully addressed this case. Now, let's assume that the intersection between these two geodesics is indeed empty. In this situation, we observe the accompanying diagram. Here, we will apply the divergence function relevant to this geodesic. By utilizing this divergence function, we can derive a bound on the length of the path, which consists of a union of geodesics.

Specifically, this path includes the segments z_1 x_3, the geodesic x_3 p, the geodesic p y_3, and the geodesic y_3 z. In the illustration, this pathway is represented as traveling from z_1 to x_3, proceeding to p, and then returning to z_2. By utilizing the divergence function and acknowledging that the distance between p, x_3, and y_3 is bounded, we can demonstrate that the distances between z_1 x_3 and z_2 y_3 are also constrained in terms of e_0.

Thus, the distance between z_1 x_3 and z_2 y_3 is bounded by e_0. Additionally, the distance z_1 x_2, as a segment of the geodesic, is a subset of the geodesic z_1 x_3. Therefore, this distance will also be bounded, as will the distance between z_2 and y_1. In conclusion, the lengths of segments z_1 x_2 and z_2 y_1 are both bounded.

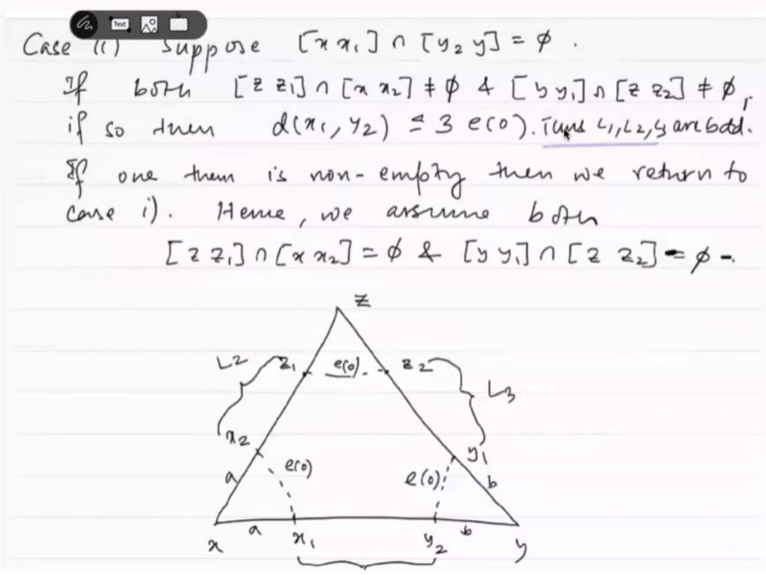
(Refer Slide Time: 17:23)



What we have demonstrated here is that this distance is indeed bounded in terms of e_0. Consequently, it follows that this triangle will be a slim triangle, with the bound directly dependent on e_0. For the point x_2, we have already identified a point here whose distance is precisely equal to e_0. Similarly, for point y_1, we have also found a corresponding point whose distance is exactly e_0.

As a result, the length is bounded in relation to e_0. Thus, we can conclude that this triangle will exhibit the characteristics of a slim triangle, and the bound will rely solely on e_0. Therefore, we have successfully addressed both case 1 and case 2.

(Refer Slide Time: 18:48)



Now, let us consider the scenario where the geodesic joining points x and x_1 does not intersect with the geodesic connecting y_2 and y. In this case, if the geodesic from z to z_1 intersects with the geodesic from x to x_2, and the geodesic from y to y_1 intersects with the geodesic from z to z_2, we can again demonstrate that the distance between x_1 and y_2 is less than or equal to 3 e_0. This establishes that the distances l_1, l_2, and l_3 are bounded.

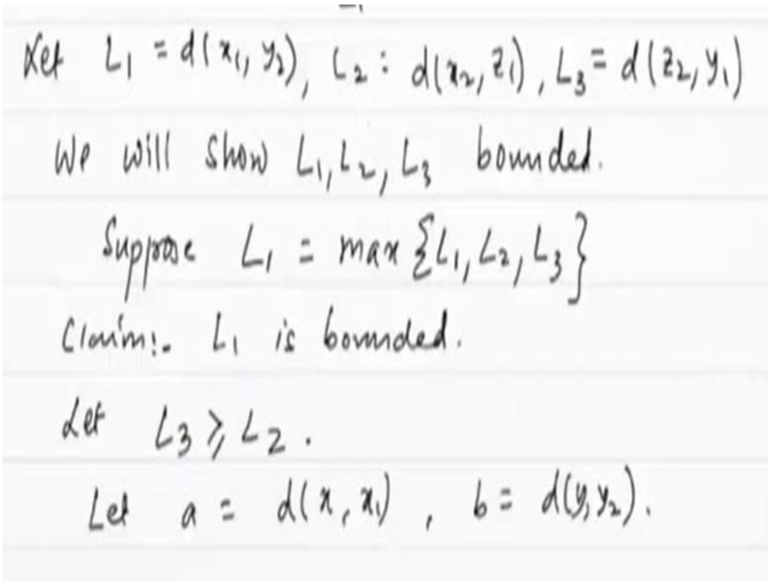
To clarify, l_1 represents the distance between x_1 and y_2, while l_2 is the distance between y_1 and z_2, and l_3 is the distance between x_2 and z. This is not particularly challenging to prove. If any of these distances is not empty, we will revert to case 1 for our analysis.

Thus, we find ourselves in a situation where the geodesic from z to z_1 does not intersect with the geodesic from x to x_2, nor does the geodesic from y to y_1 intersect with the geodesic from z to z_2. Our objective here is to demonstrate that the distances l_1, l_2, and l_3 are indeed bounded. If we can establish this, it follows that the triangle formed will be a slim triangle.

To facilitate this proof, we are categorizing the triangle into two parts: a thin part and a fat part. The section we refer to as the thin part is here, while the section we identify as the fat part lies here. Once we show that the distances l_1, l_2, and l_3 are bounded, we can conclusively demonstrate that this triangle is, indeed, a slim triangle.

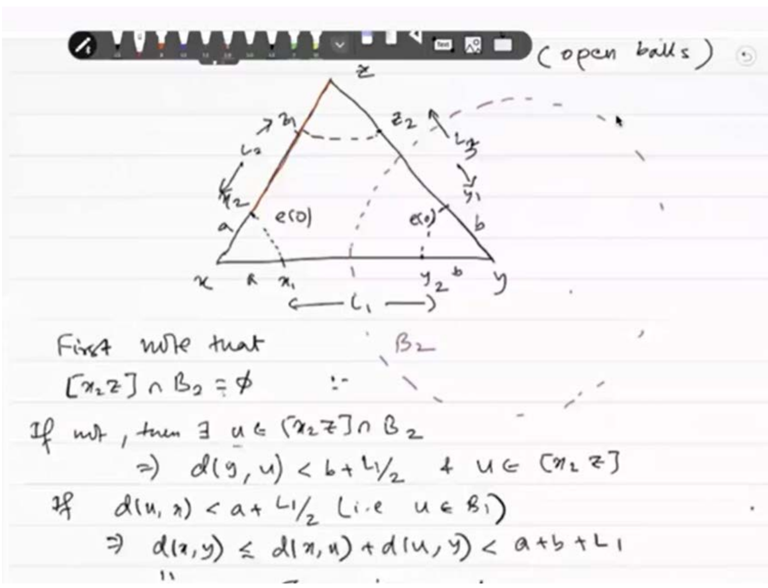
Now, let's consider l_1, which represents the maximum of the distances l_1, l_2, and l_3. If I can establish that l_1 is bounded, it will consequently prove that l_2 and l_3 are also bounded. For the sake of our discussion, let us assume that l_3 is greater than or equal to 2.

(Refer Slide Time: 21:40)



Now, let's analyze the distance from point a to point b between x_1 and y_2. This distance can be expressed as follows: the distance from a to b is equal to the distance between x_2 and b, which is equivalent to the distance between y and y_2. Additionally, we can observe that this is also equal to the distance between y_1 and x_1 added to the distance between x_1 and x_2, which is e_0, and the distance from y_2 to y_1, which is also equal to e_0. Thus, we can conclude that the overall expression holds true and reinforces our established bounds.

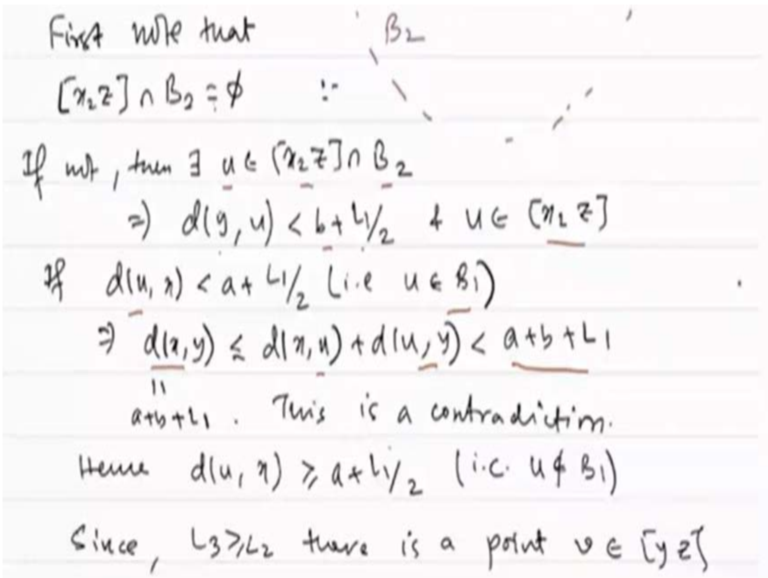
(Refer Slide Time: 22:46)



Now, let's consider a ball, which I will refer to as B_1, centered around the point x with a radius of $a + \frac{l_1}{2}$. Additionally, let’s define another ball, B_2, centered around the point y with a radius of $b + \frac{l_1}{2}$.

The first observation to make is that the geodesic connecting x to z does not intersect B_2. Now, let’s suppose, for the sake of argument, that it does intersect. If that were the case, then there would exist a point u that lies on the geodesic segment from x to z and also belongs to B_2. This point, as we have established, will be referred to as u.

(Refer Slide Time: 23:50)



This implies that the distance from y to u is less than $b + \frac{l_1}{2}$ because u lies in B_2 and also on the geodesic from x to z. Now, let's suppose that the distance from u to x is less than $a + \frac{l_1}{2}$. In this case, u would belong to B_1.

Applying the triangle inequality to the distance between x and y, we find that the distance d(x, y) can be expressed as:

$$d(x, y) \leq d(x, u) + d(u, y).$$

Here, we know that the distance d(x, u) is less than $a + \frac{l_1}{2}$ and the distance d(u, y) is also less than $b + \frac{l_1}{2}$. Thus, we can conclude:

$$d(x, y) < a + b + l_1.$$

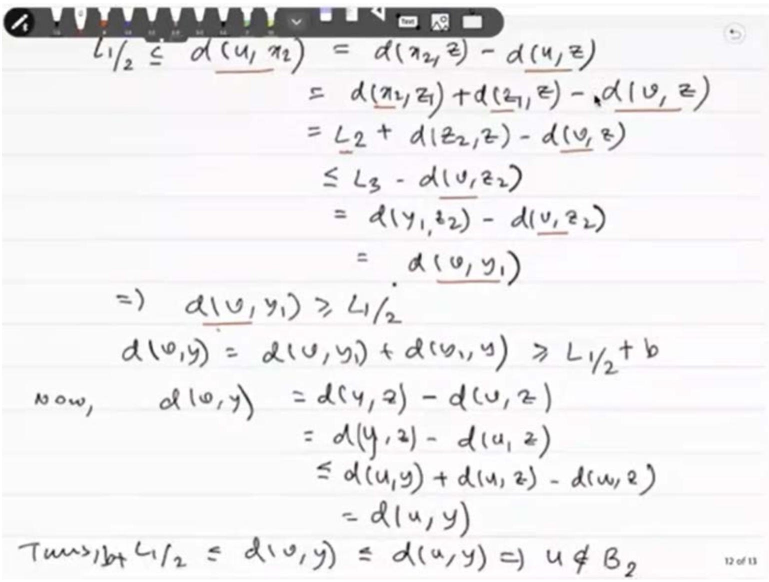
However, since x and y are connected by a geodesic, the length of this geodesic must equal a + l_1 + b. This creates a contradiction, indicating that our initial assumption is incorrect. Consequently, we must conclude that the distance from u to x is at least $a + \frac{l_1}{2}$, meaning that u does not belong to B_1.

Now, we have assumed that l_3 is greater than or equal to l_2. From this assumption, it follows that there exists a point b on the geodesic connecting y and z such that the distances from z to u and z to b are the same.

It is important to note that u lies on the geodesic, and we can identify the point b on the same geodesic such that the distances d(z, u) and d(z, b) are equal.

Finally, since $a + \frac{l_1}{2}$ is less than or equal to the distance from u to x, we observe that u is positioned such that d(u, x) = a + d(u, x_2), where u lies somewhere in that region.

(Refer Slide Time: 27:20)



Therefore, we can establish that $\frac{l_1}{2}$ is less than or equal to the distance between u and x_2. Consequently, we have the following inequality from the triangle inequality:

$$\frac{l_1}{2} \le d(u, x_2).$$

Furthermore, the distance d(u, x_2) can be expressed in terms of the distances along the geodesic, specifically as the difference between the distance from x to z and the distance from u to z. This relationship is evident from the geometric representation:

$$d(u, x_2) = d(x_2, z) - d(u, z).$$

Next, we note that the distance d(x, z) can be decomposed as follows:

$$d(x, z) = d(x, z_1) + d(z_1, z),$$

where d(x, z_1) is equal to l_2 and d(z_1, z) is simply the distance between z and z_1 minus d(v, z).

Since we know that l_2 is less than or equal to l_3, we can express this as:

$$d(x, z) = l_2 + \big(d(z, z_2) - d(v, z)\big).$$

This means that the distance d(z_2, z) minus d(v, z) leads us to conclude that:

$$d(z_2, z) - d(p, z) = -d(v, z_2).$$

Now, since l_3 is defined as the distance between y_1 and z_2 minus d(v, z_2), it follows that:

$$l_3 = d(y_1, z_2) - d(v, z_2).$$

Consequently, we have established that the distance from v to y_1 is greater than or equal to $\frac{l_1}{2}$:

$$d(v, y_1) \geq \frac{l_1}{2}.$$

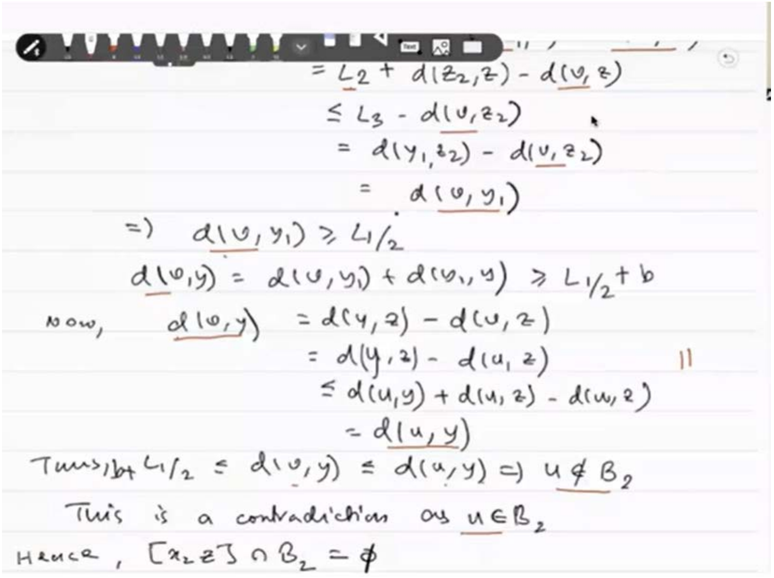
This implies that the total distance from v to y, which can be broken down into the distance from p to y plus the distance from y_1 to p, satisfies the following inequality:

$$d(v, y) = d(p, y) + d(y_1, p) \geq \frac{l_1}{2} + b.$$

Now, once again, let us examine the distances involved. The distance between v and y is less than or equal to the distance between u and y, a conclusion that follows directly from the triangle inequality. Therefore, we have established the following relationship:

$$p_1 + \frac{l_1}{2} \leq d(v, y) \leq d(u, y).$$

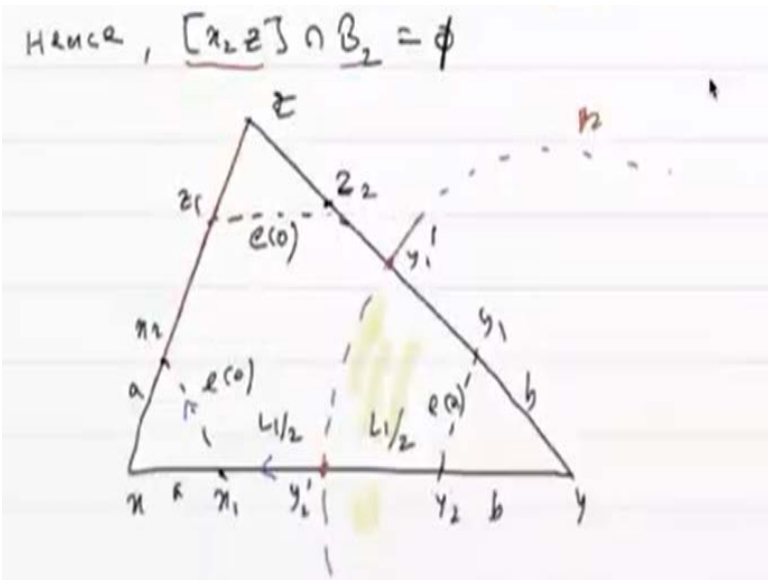
(Refer Slide Time: 29:42)



This implies that u does not belong to the ball B_2, leading us to a contradiction because we initially assumed that u is within B_2.

Thus, we can conclude that the geodesic connecting points x to z does not intersect the ball B_2. This is a significant finding: we have successfully demonstrated that this ball does not intersect the side of the geodesic from x to z.

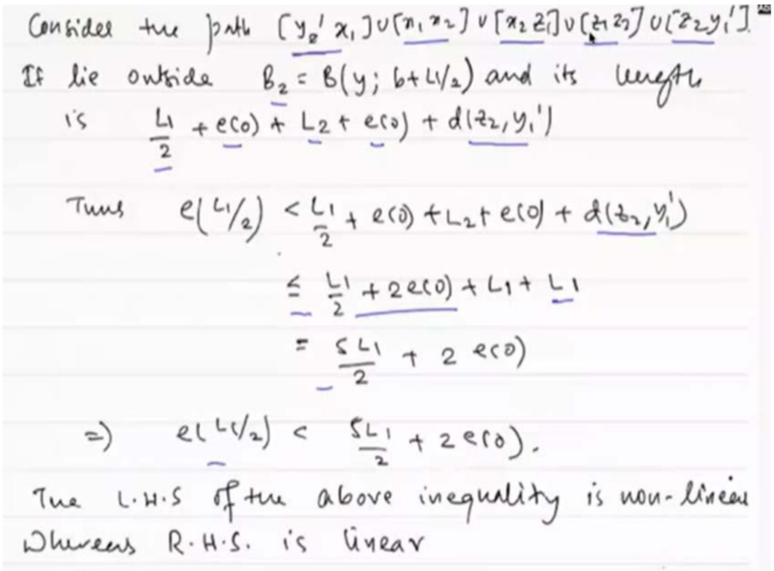
(Refer Slide Time: 31:08)



We are now in a position to apply the divergence function. Let us consider the path from y_2' to x_1. Here, y_2' represents the point where the ball B_2 intersects the side yx, while y_1' indicates the intersection with the side yz.

The path we will take consists of several segments: first, we will traverse the geodesic from y_2' to x_1, which is depicted here. Next, we will move from x_1 to x_2. After reaching x_2, we will continue along the path to z_1, followed by the segment from z_1 to z_2. Finally, we will conclude our journey by moving from z_2 to y_1'. This sequence of paths has been clearly outlined here for clarity.

(Refer Slide Time: 32:12)



Let us begin by taking the geodesic from y_2' to x_1, followed by the segment from x_1 to x_2. Next, we will unite this with the geodesic from x_2 to z_1, and then combine it with the segment from z_1 to z_2. Finally, we will include the path from z_2 to y_1'.

This entire path lies outside the ball B_2, and its total length can be expressed as $\frac{l_1}{2} + e_0 + l_2 + e_0 + \text{distance}(z_2, y_1')$. Given that the distance between z_2 and y_1' is $\frac{l_1}{2}$, we have:

$$\text{Distance}(x_1, y_2) = l_1,$$

which leads to the conclusion that the total distance is $\frac{l_1}{2}$.

In this context, we note that the individual lengths are as follows: the distance from x_1 to y_2 is $\frac{l_1}{2}$, the distance from the geodesic to z_1 is e_0, the distance to z_2 is l_2, another distance of e_0, and finally, the distance from z_2 to y_1' is $\frac{l_1}{2}$.

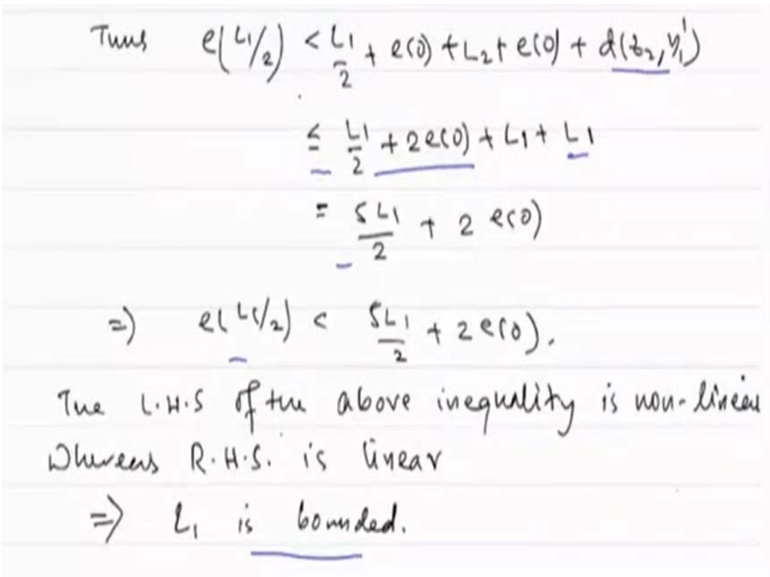
Now, let's apply the definition of the divergence function e. For $e\left(\frac{l_1}{2}\right)$, it is important to revisit the earlier diagram to visualize that this distance is indeed $\frac{l_1}{2}$. Thus, we can infer that $e\left(\frac{l_1}{2}\right)$ will be less than the length of the path we have established.

Furthermore, since the total path length is inherently less than or equal to l_1 (as l_1 represents the maximum), we can derive the inequality:

$$e\left(\frac{l_1}{2}\right) < 5\left(\frac{l_1}{2}\right) + 2e_0.$$

So, we have established that $e\left(\frac{l_1}{2}\right)$ is indeed less than $5\left(\frac{l_1}{2}\right) + 2e_0$.

(Refer Slide Time: 34:45)



The left-hand side of the above inequality is non-linear, while the right-hand side is linear. This distinction establishes that l_1 is bounded in terms of e_0. Consequently, both l_2 and l_3 will also be bounded, since l_1 represents the maximum of l_1, l_2, and l_3.

Therefore, what we have demonstrated is that l_1, l_2, and l_3 are indeed bounded in terms of e_0. As a result, we can conclude that this triangle will be classified as a slim triangle. I will now stop. Thank you!