An Introduction to Hyperbolic Geometry

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Module No. 07

Lecture No. 32

Exponential Divergence of Geodesics: A Study of Hyperbolic Metric Spaces

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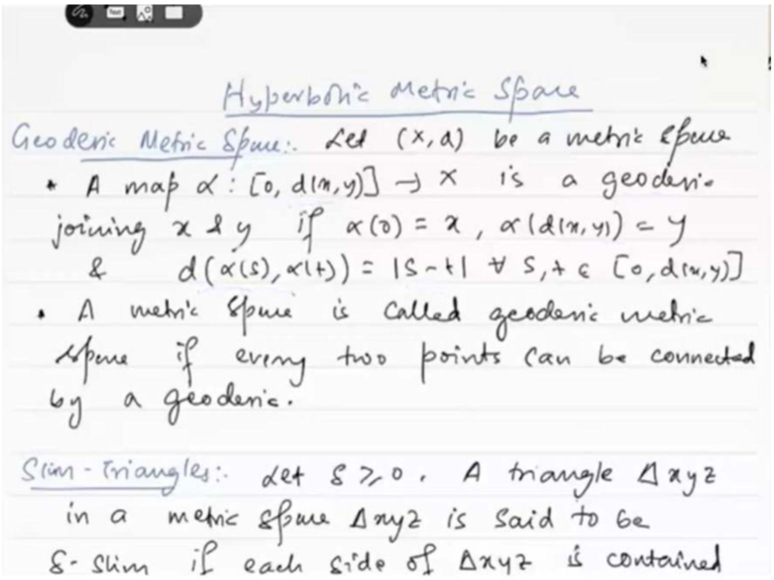

Hello! In this lecture, we will revisit the definition of hyperbolic metric spaces, specifically in terms of slim and thin triangles. Additionally, we will introduce the concept of divergence of geodesics and demonstrate that in hyperbolic metric spaces, geodesics diverge exponentially.

Let us begin. We start with the concept of a geodesic metric space. A metric space X is classified as a geodesic metric space if every pair of points in X can be connected by a geodesic. So, what exactly do we mean by a geodesic?

A map α from the closed interval [0, d] to X is considered a geodesic if it satisfies the following conditions: α(0) = x, α(d) = y, and the distance between α(s) and α(t) is given by |s - t| for all s, $t \in [0, d]$.

We have defined slim triangles, so let us now introduce a non-negative number δ. A triangle Δ x y z in a metric space is said to be δ-slim if each side of Δ x y z is contained within a closed δneighborhood of the union of the other two sides of the triangle. This concept has been previously established.

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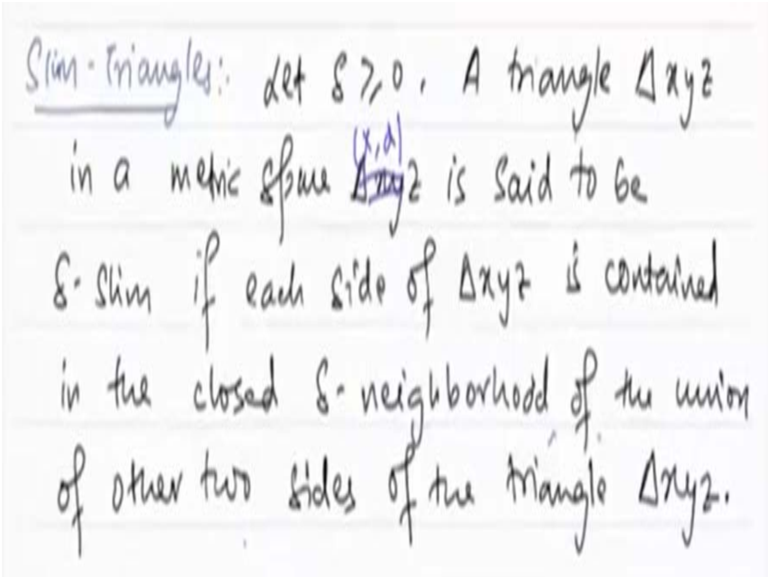

Now, what do we mean by a thin triangle?

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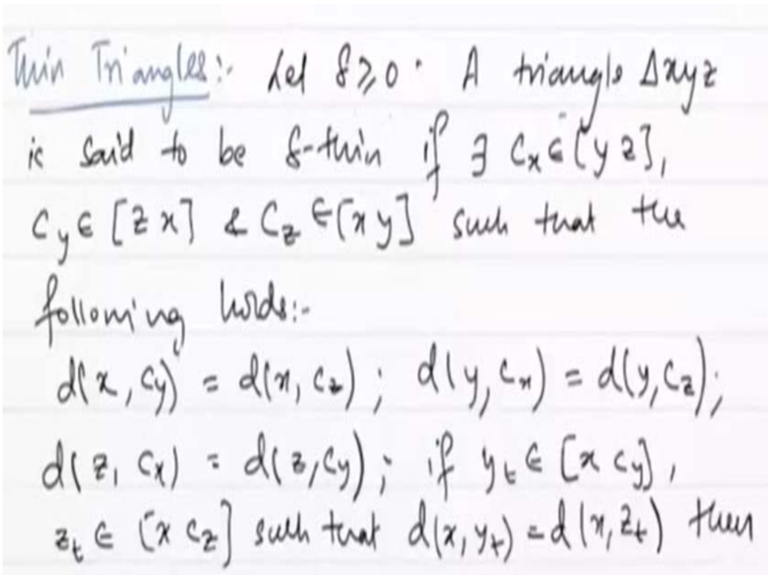

Let us consider a non-negative number δ. A triangle Δ x y z is said to be δ-thin if there exist points c_x, c_y, and c_z such that c_x belongs to the line segment y z, c_y belongs to the line segment z x, and c_z belongs to the line segment x y, satisfying the following conditions:

- The distance between x and c_y is equal to the distance between x and c_z.
- The distance between y and c_x is equal to the distance between y and c_z.
- The distance between z and c_x is equal to the distance between z and c_y.

Furthermore, if y_t belongs to the geodesic connecting x and c_y, and z_t belongs to the geodesic connecting x and c_z, then the distance between x and y_t is equal to the distance between x and z_t.

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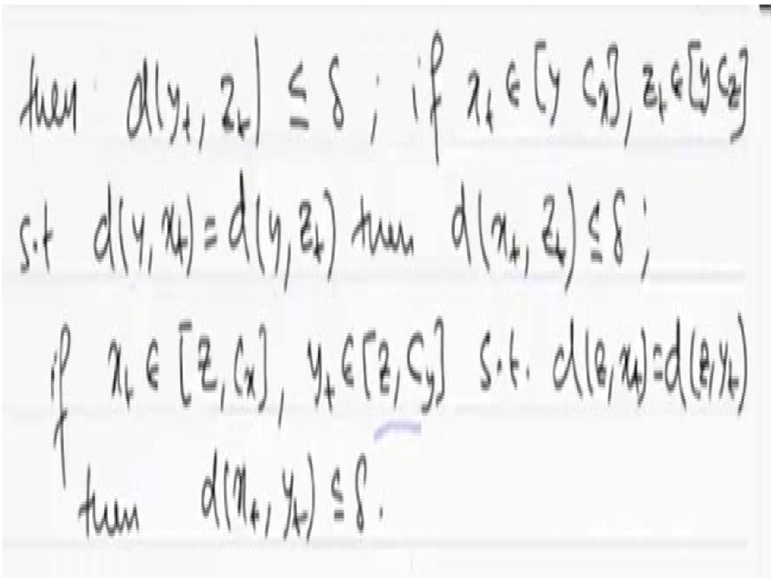

Thus, the distance between y_t and z_t is less than or equal to δ, and similarly for the other sides of the triangle. If x_t belongs to the geodesic connecting y and c_x, and z_t belongs to the geodesic connecting y and c_z, such that the distance from y to x_t is equal to the distance from y to z_t, then the distance between x_t and z_t will also be less than or equal to δ.

Additionally, if x_t belongs to the geodesic connecting z and c_x, and y_t belongs to the geodesic connecting z and c_y, ensuring that the distance from z to x_t is equal to the distance from z to y_t, then it follows that the distance between x_t and y_t is also less than or equal to δ. We covered this analysis in the previous class and successfully proved this theorem.

If a triangle is classified as thin, it consequently qualifies as slim. Thus, when we consider a triangle to be δ-thin, it will also be δ-slim. This is the essence of this concept.

Now, the converse of this statement can be articulated as follows: in a metric space, if all triangles are δ-slim, then there exists a δ', which depends on δ, such that all triangles are δ'-thin.

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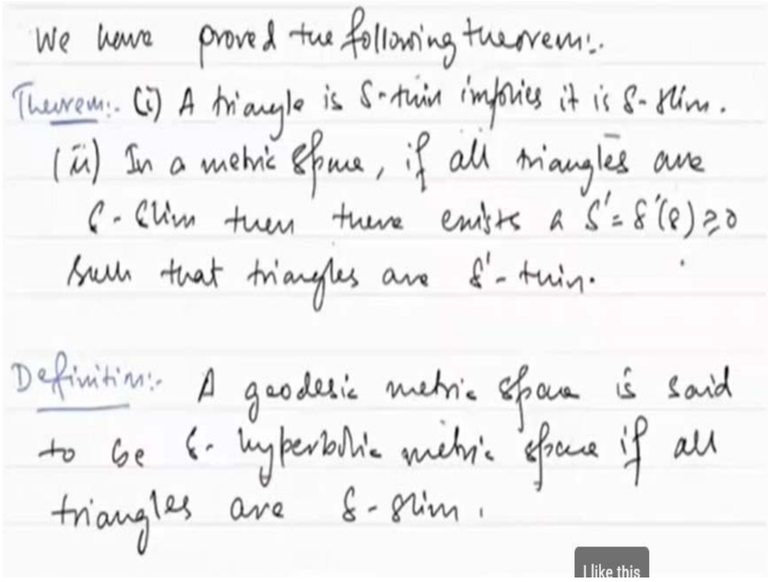

Now, let's define what it means for a metric space to be hyperbolic. A geodesic metric space is termed a δ-hyperbolic metric space if all its triangles are δ-slim. This implies that in a δ-hyperbolic metric space, there exists a δ' such that all triangles are δ'-thin.

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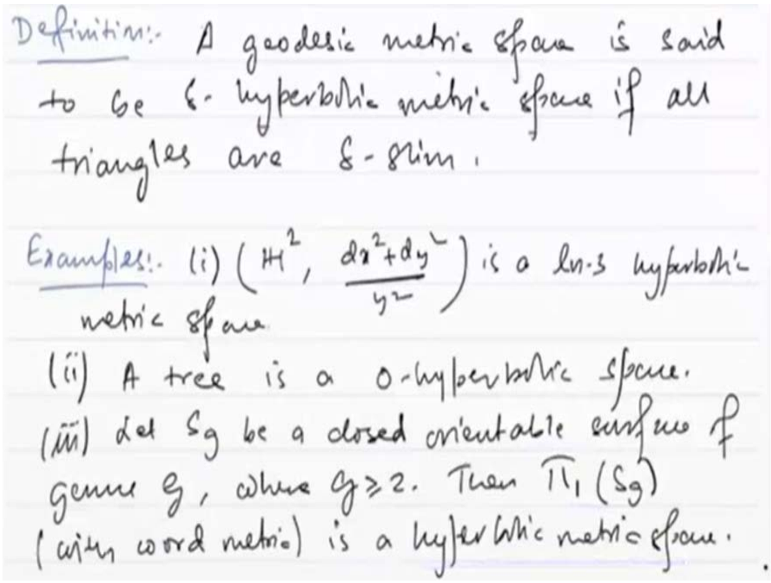

We have already encountered some examples in our previous discussions. For instance, consider the upper half-plane equipped with the hyperbolic metric; this serves as an example of a log(3)-hyperbolic metric space. Additionally, it's important to note that a surface with genus g = 0 is a hyperbolic metric space as well, which we will explore in greater detail later.

Now, let's turn our attention to a third example, which we will also examine further down the line. If we take S_g to be a closed orientable surface of genus g, where g is greater than or equal to 2, then the fundamental group of this genus g surface can be classified as a hyperbolic metric space. We will provide a proof for this assertion in the upcoming sections.

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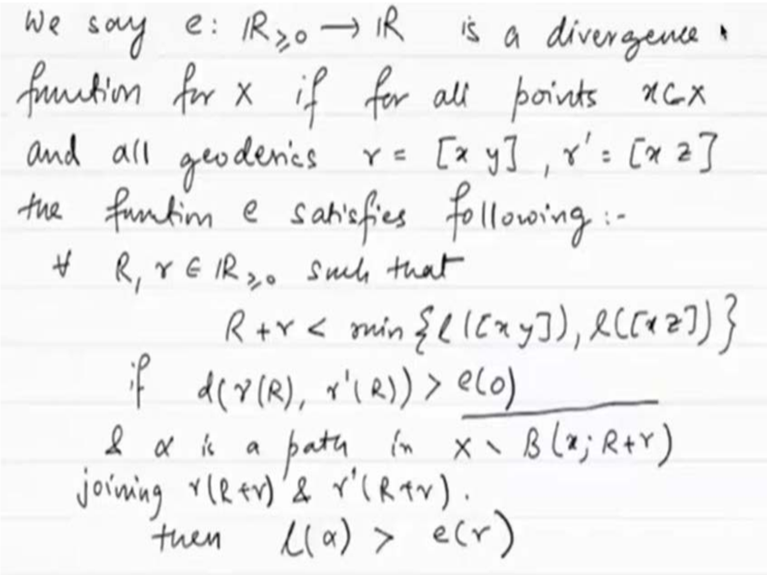

Now, let’s delve into the definition of the divergence function. We define a function $e: R_{\geq 0} \rightarrow R$ to be a divergence function for a metric space X if it meets the following criteria: For every point x in this symmetric space and for all geodesics γ connecting x to y and γ' connecting x to z, the function e must satisfy the condition that for all R and r belonging to $R_{\geq 0}$, the sum R + r is less than the minimum of the lengths of the geodesics $\overline{xy} and \overline{xz}$.

Furthermore, we assume that the distance between γ(R) and γ'(R) is greater than zero. If α is a path that lies in the closure of the complement of a ball around x with radius r + R, then α must lie in the space connecting the two points γ(R + r) and γ'(R + r). In this case, the length of the path α must exceed e(r).

Let me illustrate this concept with a diagram. Here is point x, from which two geodesics emerge. I’ve labeled one geodesic as γ and the other as γ'. The point γ(r) lies on the geodesic γ, while γ'(r) is located on the geodesic γ'. The distance from x to γ(r) is r, which is, interestingly, equal to the distance from x to γ'(r).

Now, what we have done is take points γ(R + r) on each geodesic. Next, we consider a ball of radius R + r centered at x. This is our ball, and what we will do now is take a path α that lies outside this ball. Importantly, α is a path situated in the closure of the complement of this ball.

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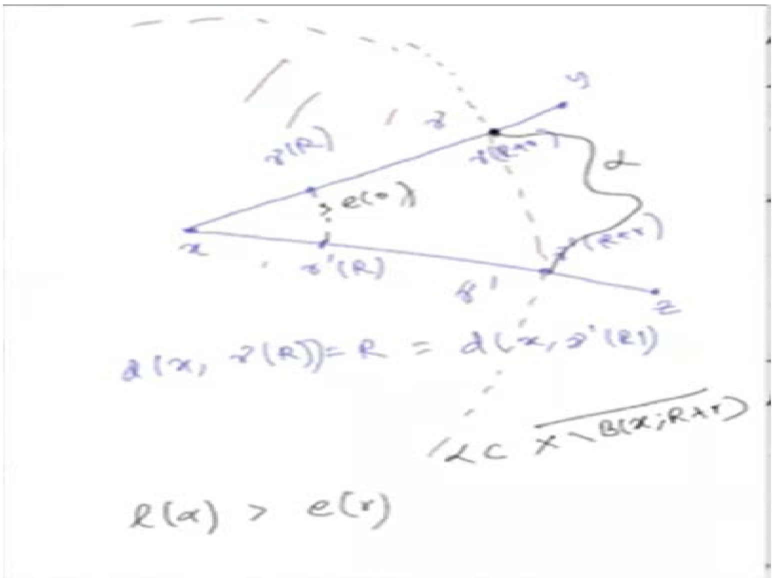

For a function e to be classified as a divergence function, we also need to assume that the distance between γ(r) and γ'(r) is greater than e(0). We define e as a divergence function if the length of path α exceeds e(r). Our goal is to demonstrate that this function e is exponential if x is a delta hyperbolic metric space.

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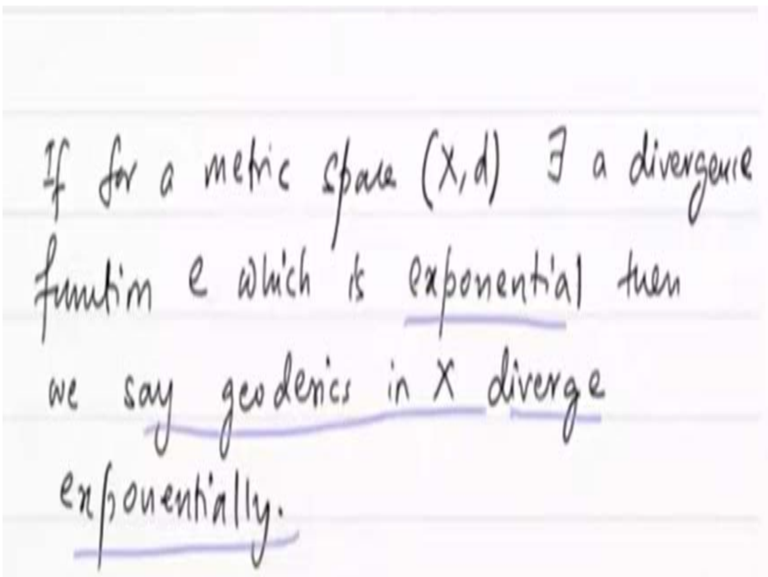

In the context of a metric space x, if there exists a divergence function e that is exponential, we can then conclude that the geodesics within this space x diverge exponentially.

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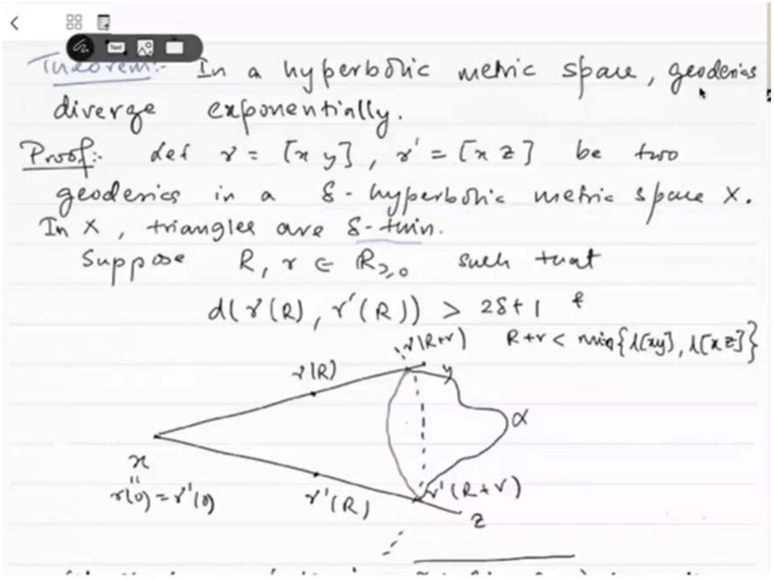

Here is my theorem: in a hyperbolic metric space, the geodesics diverge exponentially. To establish this, we need to identify a divergence function that exhibits exponential behavior. So, let's explore how to prove this statement.

Consider two geodesics, γ connecting points x to y and γ' connecting x to z in a delta hyperbolic metric space x. Given that x is a delta hyperbolic metric space, we can confidently assert that the triangles formed within this space are delta thin.

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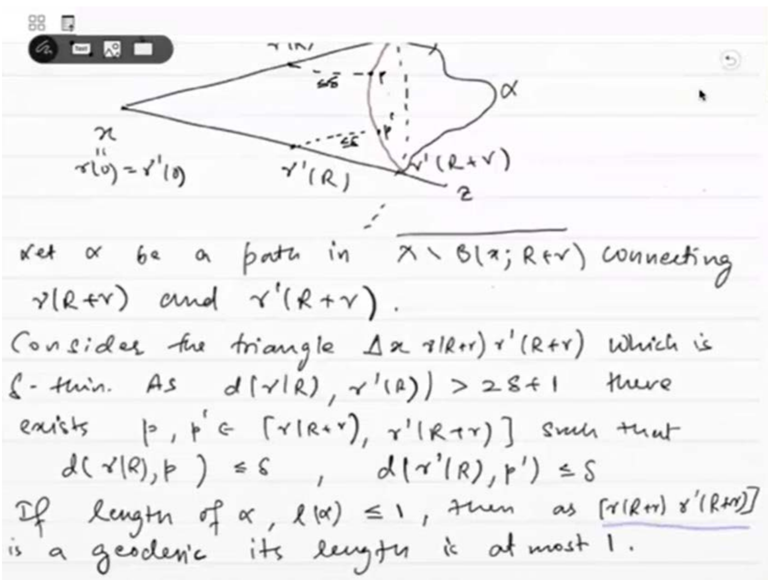

Now, let us consider two non-negative numbers, R and r, such that the distance between γ(r) and γ' (r) is greater than 2δ + 1. Here, we have our two geodesics, and we are assuming this distance is indeed greater than 2δ + 1. Additionally, we also assume that R + r is less than the minimum of the lengths of $\overline{xy}$ and $\overline{xz}$.

Next, we take the points γ(r + r), which lies on $\overline{xy}$, and γ'(r + r), which lies on $\overline{xz}$.

Now, let us define a path α as a path that resides in the closure of the complement of a ball with radius R + r centered around x, connecting the points γ(r + r) and γ'(r + r). We will join these two points, γ(r + r) and γ'(r + r), by a geodesic in x, where we are treating x as a geodesic metric space.

Now, consider the triangle formed by these points. This triangle is known to be delta thin. Because these two points are at a distance apart, we can find points p on the geodesic connecting γ(r + r) and γ'(r + r) such that the distance between them is less than or equal to δ.

If the length of α is less than or equal to 1, then, given that this is a geodesic, its length will also be at most 1.

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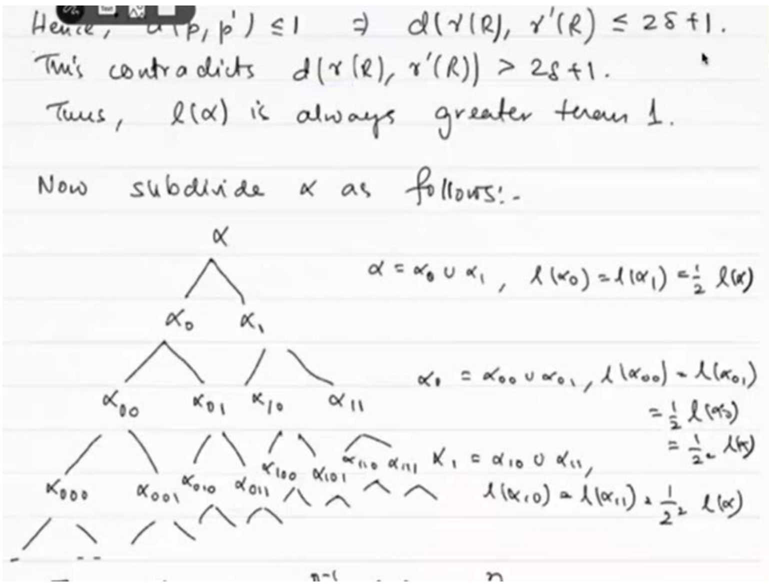

Thus, the distance between points p and p' will also be less than or equal to 1. Consequently, the distance between γ(r) and γ'(r) must also be less than or equal to 2δ + 1. However, if this distance is indeed less than or equal to 1, it follows that the distance between γ(r) and γ'(r) would also be less than or equal to 2δ + 1. This situation leads to a contradiction. Therefore, we conclude that the length of α cannot be less than or equal to 1; rather, it must always be greater than 1.

This is precisely why we have established the distance between γ(r) and γ'(r) to be greater than 2δ + 1.

Now, let’s discuss the next steps: we will subdivide the path α in the following manner. The first subdivision will involve expressing α as the union of two segments: α_0 and α_1, such that the lengths of both α_0 and α_1 are equal to half the length of α.

In other words, we divide α into two equal parts. This means we will obtain α_0 and α_1 in such a way that the length of α_0 is equal to the length of α_1, both being half of the total length of α. This process continues; you will take α_0 and again subdivide it into two equal parts, while also doing the same for α_1.

Continuing this subdivision process leads us to a natural number n such that the length of α is greater than n - 1 but less than or equal to 2^n. This implies that $\frac{1}{2}$ is less than $\frac{1}{2^n} \times$ length of α, and this is less than or equal to 1.

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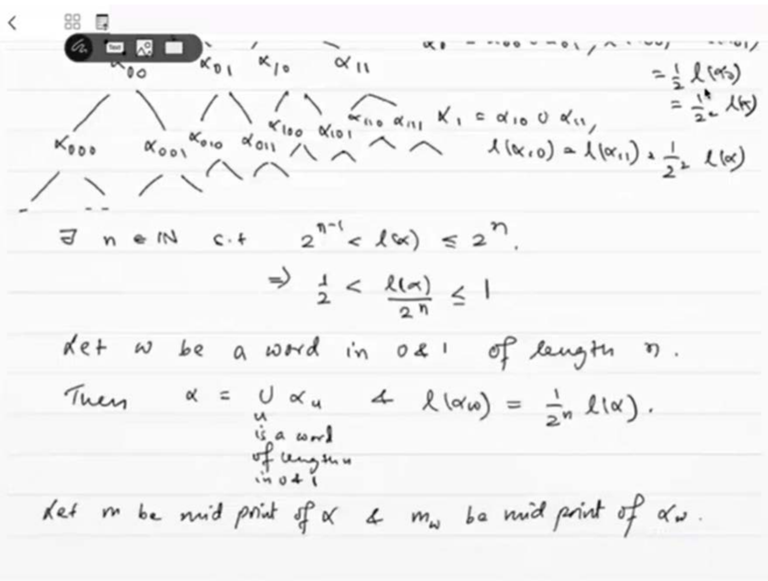

We have divided the path α into two equal parts, and then we further subdivide each segment into equal parts, continuing this process until we reach the endpoint. This leads us to the situation where we have established that $\frac{1}{2}$ is less than the length of α divided by 2^n, which is also less than or equal to 1.

Now, let's introduce w as a word formed from the set {0, 1}. For example, if we take n = 3, we could have the word 010, which is a valid word of length 3, or 011, which is also a word of length 3. So, let us consider w to be a word composed of 0 and 1 with a length of n.

Consequently, α can be expressed as the union of segments α_u, where u is a word of length n in the set {0, 1}. The length of each sub-word or segment α_w will then be equal to $\frac{1}{2^n}$ times the length of α.

Next, let's define m to be the midpoint of the path α and m_w to be the midpoint of the segment α_w. It is important to note that w is indeed a word of length n.

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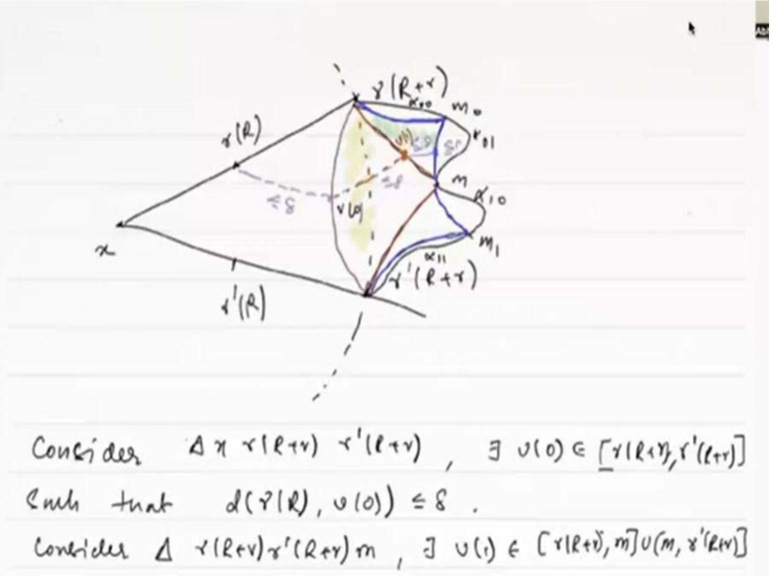

Here's a visual representation of the scenario we're discussing. This is the path α, and we have identified the midpoint m. The point m_0 represents the midpoint of the segment α_{01}, while m_1 serves as the midpoint of α_{10}. We will continue this process of subdivision.

Now, let's connect the points γ_{r+r} and m using a geodesic, and similarly, connect γ'_{r+r} to m via another geodesic. Next, we focus on the point m_0 and again join γ_{r+r} to m_0 with a geodesic. We will also join m_0 to m using a geodesic, and finally connect m_2 to m_1 by a geodesic, along with γ'_{r+r} to m_1.

As we continue this process, we will have multiple triangles to consider. Let's begin with the triangle formed by the points γ_{r+r}, γ'_{r+r}, and m. This triangle is indeed delta thin, and because the distance between γ_r and γ'_r is greater than $2\delta + 1$, there will exist a point on this geodesic segment. We will designate this point as p_0, where the distance between γ_r and p_0 is less than or equal to δ.

Next, let’s analyze the triangle composed of the points γ_{r+r}, γ'_{r+r}, and m. In this scenario, there will exist a point v_1 along the union of these two sides such that the distance between v_0 and v_1 will also be less than or equal to δ. In our diagram, the point v_1 is represented as p_1, and its distance from v_0 satisfies the condition of being less than or equal to δ.

Now, if we suppose that v_1 lies on the geodesic between γ_{r+r} and m, we can further analyze the triangle formed by these points. For v_1, we will again find a point, either on the geodesic joining γ_{r+r} to m_0 or the geodesic from m_0 to m, such that the distance from p_1 to this point is less than or equal to δ. Let’s designate that point as v_2, and we can place v_2 on the geodesic that connects m and m_0, as noted in our description.

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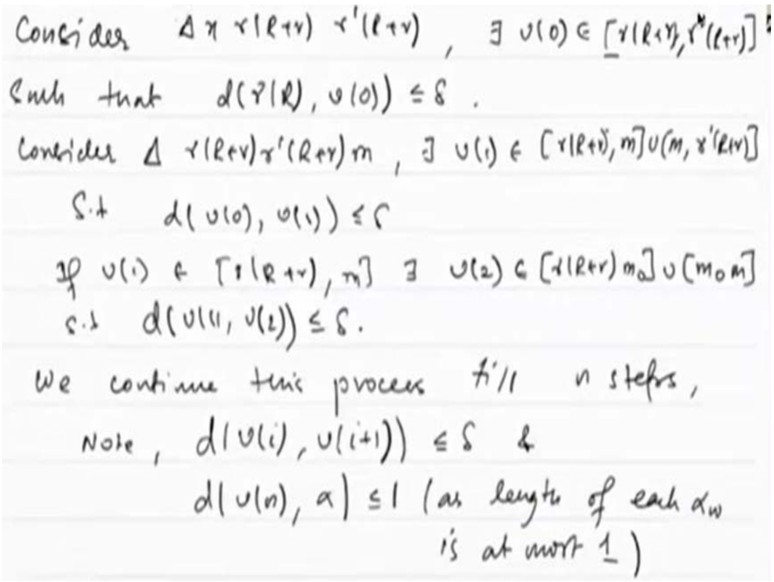

If p_1 lies on this geodesic, there exists a point p_2 that belongs to the union of these two geodesics, ensuring that the distance between v_1 and v_2 is less than or equal to δ. We will continue this iterative process for n steps.

Now, it's important to note that the distance between v_i and v_{i+1} will always be less than or equal to δ. By the time we reach the n-th step, the distance between v_n and α will be less than or equal to 1, since the length of each segment α_w is at most 1. Specifically, the length of α_w is equal to the length of α divided by 2^n, which means it is indeed less than or equal to 1.

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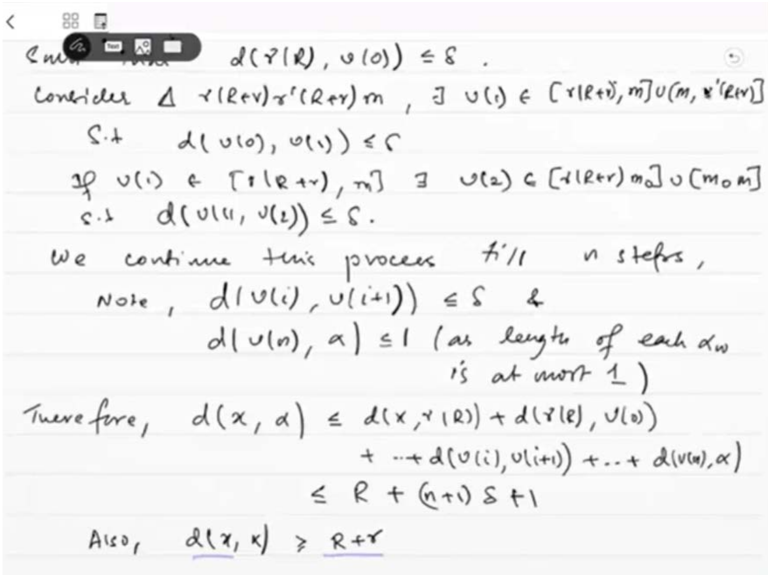

Now, let’s analyze the distance from x to the path α. This distance will be less than or equal to the application of the triangle inequality. Specifically, the distance from x to the path α can be expressed as the sum of distances:

$$\text{distance}(x, \alpha) \le \text{distance}(x, \gamma_r) + \text{distance}(\gamma_r, v_0) + \text{distance}(v_i, v_{i+1}) + \cdots + \text{distance}(v_n, \alpha).$$

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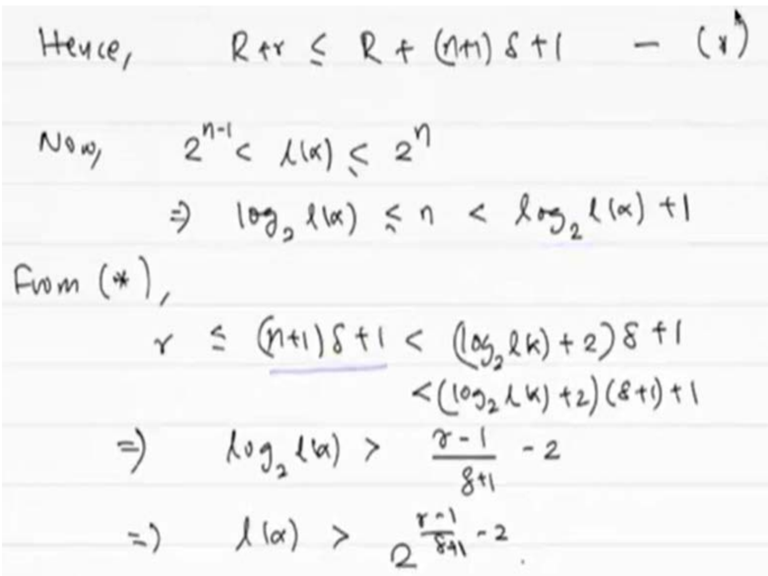

The first term, distance(x, γ_r), is equal to r because γ_r is a geodesic. As we have n + 1 terms here, and each of these segments is less than or equal to δ, we can say that the total distance is:

$$\text{distance}(x, \alpha) \leq r + (n + 1) \times \delta.$$

It follows that this distance is less than or equal to 1. Additionally, it's essential to observe that since α lies outside the ball of radius R + r centered at x, the distance between x and α must be greater than or equal to R + r.

Therefore, based on these two inequalities, we have R + r ≤ some expression. It's also important to note that $2^{n-1} <$ length of $\alpha \leq 2^n$. This implies that if we take the logarithm base 2 of the length of α, we find:

$$\log_2(\text{length of } \alpha) \leq n < \log_2(\text{length of } \alpha) + 1.$$

From this relationship, we can deduce that $r \leq (n + 1) \times \delta + 1$. Now, what can we conclude from this inequality?

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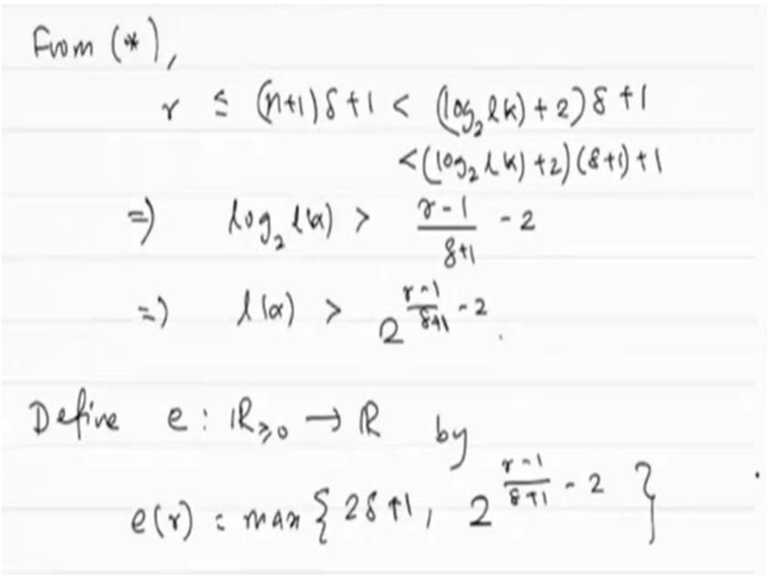

Thus, we conclude that log(length of α) is less than log(length of α) $+ 2 \times \delta + 1$, leading us to find that the length of α is at least $10 \cdot 2^{r-1}/(\delta + 1) - 2$.

Now, let's define the function e from $R_{\geq 0}$ to R in the following manner:

$$e(r) = \max\left(2\delta + 1, \frac{2^{r-1}}{\delta + 1} - 2\right).$$

This function e serves as a divergence function, and importantly, it is classified as an exponential divergence function.

Consequently, we have demonstrated that geodesics in hyperbolic metric spaces exhibit exponential divergence. In our next lecture, we will prove that if geodesics diverge exponentially in a geodesic space, then that metric space must indeed be a hyperbolic metric space. I will conclude my presentation here.