An Introduction to Hyperbolic Geometry
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Hyperbolic Metric Spaces: Properties of Slim Triangles and Geodesics

In a hyperbolic plane, the triangles are slim, meaning that if you take any triangle in this plane, each side of the triangle lies within a log 3 neighborhood of the union of the other two sides. This is a characteristic of triangles in hyperbolic geometry.

Now, we'll extend this concept to a more general setting, specifically exploring this property in a general hyperbolic metric space. We begin with a metric space where any two points can be connected by a geodesic. If, in this metric space, all triangles exhibit a condition called "δ-slimness" (for a pre-chosen non-negative number δ), meaning that each side of the triangle is close to the union of the other two sides, then we refer to such a metric space as a hyperbolic metric space.

In the following discussion, we will introduce and describe several different definitions and properties of hyperbolic metric spaces. Let's begin with these foundational concepts.

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Let's first begin with the definition of a geodesic in a metric space. Consider two points in a metric space, denoted as (X, d), where X is the set of points and d is the distance function. A path α is defined from the closed interval [0, d(x,y)] to X, where d(x,y) is the distance between the points x and y. The path α is said to be a geodesic between x and y if the following conditions hold:

1. α(0) = x, and α(d(x,y)) = y,

2. The distance between α(s) and α(t) is |s - t| for all s, t belonging to the interval [0, d(x,y)].

This is the formal definition of a geodesic.

For example, let's consider the upper half-plane model with hyperbolic geometry. If we take the upper half-plane and define the path α from the closed interval $[0, \log(b/a)]$ to H^2 in the following way: $\alpha(s) = (0, ae^s)$, then the distance between α(s) and α(t) will be $\log\left(\frac{ae^s}{ae^t}\right)$, which simplifies to |s - t|. Hence, according to this definition, α is indeed a geodesic.

In our earlier lectures, we defined a geodesic as a length-minimizing path. If we consider the arc length parameterization of such a path, it will correspond to this formal definition of a geodesic.

Another example can be found in Euclidean space. If you take any straight line, it will also be a geodesic. In order to apply this definition of a geodesic, you simply need to parameterize the path by its arc length.

Now, let's move on to defining what we mean by a geodesic metric space.

A metric space (X, d) is said to be a geodesic metric space if, for every pair of points in X, there exists a geodesic connecting them. For instance, if we take the upper half-plane equipped with the hyperbolic metric, it qualifies as a geodesic metric space. Similarly, the Euclidean plane is also a geodesic metric space, where any two points can be connected by a straight line, which is the geodesic in this case.

Now, let's move on to the concept of slim triangles. Consider a non-negative number δ, and by "triangle" here, we mean a geodesic triangle. A geodesic triangle, denoted as Δ xyz, is said to be δ-slim if each side of the triangle lies within the closed δ-neighborhood of the union of the other two sides.

To visualize this, consider a geodesic triangle, where each side is contained within a δ-neighborhood of the other two sides. For example, if we take the side xy, it must be contained within the δ-neighborhood of the union of the sides xz and yz. This means that for any point p on the side xy, there exists a point q on either the side xz or yz, such that the distance between p and q is less than or equal to δ. The same holds true for the other sides of the triangle as well. If we take the side yz, it is also contained in the δ-neighborhood of the union of the other two sides, and similarly for the side xz.

This is the idea behind a slim triangle.

Now, based on this concept, let us proceed to define what we mean by a hyperbolic metric space.

Let's consider a non-negative number δ. A geodesic metric space (X, d) is said to be δ-hyperbolic if all the triangles in this metric space are δ-slim. In simpler terms, every geodesic triangle in this space adheres to the δ-slim condition, where each side of the triangle lies within the δ-neighborhood of the other two sides.

Additionally, a geodesic metric space (X, d) is called a hyperbolic metric space if it is δ-hyperbolic for some non-negative number δ.

For a concrete example, consider the upper half-plane equipped with the hyperbolic metric. This space is a log 3-hyperbolic metric space, meaning all triangles in this space are log 3-slim.

Another example is a tree. What do we mean by a tree in this context? A tree is a graph that has no circuits, or in other words, it contains no loops. The metric on a tree is defined by taking each edge to be isometric to the unit interval. This induces a metric d on the tree such that the distance between any two distinct vertices v and w is a natural number, corresponding to the number of edges connecting them. Thus, the tree also forms a hyperbolic metric space.

For example, consider a tree. Suppose we take three points: let's denote them as x, y, and z. To prove that this tree forms a hyperbolic metric space, we need to find a δ value for which the tree

is δ-hyperbolic.

Now, if we examine the geodesics connecting these three points, we must note that the edges of the tree are of unit length. The distance between any two points is simply the number of edges between them, as each edge is isometric to the unit interval.

Let's take two points p and q on the tree. There exists a unique path connecting p and q with the minimum possible length. How do we determine this path? From p, we travel along the edges to a common vertex, and from there continue to q. The distance between p and q is calculated as follows: Let the distance from p to the vertex be l_1, and from this vertex to q be l_2. Since the tree has unit-length edges, the total distance between p and q is $l_1 + 2 + l_2$.

Now, consider the three points x, y, and z. First, we look at the geodesic connecting x and y. It simply goes from x to the shared vertex and then to y. Similarly, there's a unique geodesic connecting x to z and another connecting y to z.

In this case, the structure of the geodesics forms something resembling a tripod. If we denote the common vertex as p, the geodesic triangle formed by x, y, and z is essentially a tripod. So, the triangle in this tree looks like a tripod where each side is contained within the union of the other two sides.

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For instance, the side xy is contained within the union of the other two sides, xz and yz. The same applies for the other sides of the triangle. Hence, the triangle xyz is 0-slim, meaning that

trees are 0-hyperbolic metric spaces.

Now, let me explain the concept of thin triangles.

Let's start by considering a geodesic triangle, $\Delta\ xyz$, in a metric space. Our goal here is to identify specific points on each side of this triangle. Let's denote these points as c_x, c_y, and c_z, where c_x is on the side yz, c_y is on the side xz, and c_z is on the side xy.

The objective is to find these points such that certain distances are equal. Specifically:

- The distance from x to c_y should be equal to the distance from x to c_z,
- The distance from z to c_y should be equal to the distance from z to c_x,
- The distance from y to c_z should be equal to the distance from y to c_x.

These points help us define what is known as a thin triangle.

Now, let's formalize this in a proposition. Consider a triangle $\Delta\ xyz$ in a metric space (X, d). There exist points c_x on the side yz, c_y on the side xz, and c_z on the side xy such that:

- The distance from x to c_y is the same as the distance from x to c_z,
- The distance from y to c_x is the same as the distance from y to c_z,
- The distance from z to c_x is the same as the distance from z to c_y.

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Now, why do such points exist? What is the reasoning behind this?

We will discuss the proof next, explaining why these points c_x, c_y, and c_z must exist for any geodesic triangle in a metric space.

Let's go ahead and define the key distances and work through the solution, leveraging some linear algebra to determine the points c_x, c_y, and c_z.

Let a, b, and c represent the distances between the points we are interested in. Specifically:

- Let a = $d(x, c_y)$, the distance between x and c_y,
- Let b = $d(z, c_y)$, the distance between z and c_y,
- Let c = $d(y, c_x)$, the distance between y and c_x.

Our goal is to find the values of a, b, and c.

Now, let's write the conditions that govern these distances:

1. a + b = d(x, z),

2. b + c = d(z, y),

3. c + a = d(y, x).

These three equations represent the relationships between the distances in the geodesic triangle. To solve for a, b, and c, we can use linear algebra.

From the first equation, a + b = d(x, z), we have the sum of the distances between x and z. From the second equation, b + c = d(z, y), and the third equation, c + a = d(y, x), we now have a system of three equations that can be solved.

Using linear algebra, we find:

- $a = \frac{1}{2}\big(d(x,z) + d(x,y) - d(y,z)\big),$
- $b = \frac{1}{2}\big(d(x,z) + d(z,y) - d(x,y)\big),$
- $c = \frac{1}{2}\big(d(y,z) + d(x,y) - d(x,z)\big).$

These values of a, b, and c satisfy the conditions, meaning that the points c_x, c_y, and c_z must exist in the geodesic triangle. This process shows how we can determine the exact positions of these points using the distances between the vertices of the triangle.

Let us revisit the triangle x, y, z and establish a clearer understanding of the relationships between the points. Consider the points c_x, c_y, and c_z, such that the distance between x and c_y is the same as the distance between x and c_z. Likewise, the distance between y and c_z is equal to the distance between y and c_x, and similarly for z.

Now, notice that there exist points x_t on the geodesic joining x and c_y, and another point x_t' on the geodesic joining x and c_z, such that the distance from x to x_t is equal to the distance from x to x_t'. Both of these distances are the same, and x_t lies on the segment between x and c_y, while x_t' lies on the segment between x and c_z. Let the distance from x to x_t (and hence to x_t') be denoted by t. Clearly, $t \leq d(x, c_y)$, which is also equal to d(x, c_z).

Similarly, we can find points on the other sides of the triangle. On the geodesic joining y and c_z, there is a point y_s, and on the geodesic joining y and c_x, there is a point y_s', such that the distance from y to y_s is equal to the distance from y to y_s', denoted as s. Again, $s \leq d(y, c_z)$.

For the vertex z, we similarly find points z_r and z_r' on the geodesics joining z to c_x and c_y, respectively, where the distances from z to these points are equal, denoted by r, and $r \leq d(z, c_x)$.

A triangle is said to be δ-thin if the following conditions hold: for all points x_t, x_t', y_s, y_s', z_r, z_r', the distance between x_t and x_t', y_s and y_s', and z_r and z_r' is less than or equal to δ. That is,

$$d(x_t, x_t') \leq \delta, \quad d(y_s, y_s') \leq \delta, \quad d(z_r, z_r') \leq \delta.$$

If these conditions are met, we refer to the triangle as being δ-thin. In terms of geometry, this implies that the points on the geodesics between the vertices are close to each other, with the distance bounded by δ, indicating that the triangle is thin.

Therefore, the thinness of a triangle is characterized by the property that, for any pair of corresponding points on the geodesics, the distance between them is always less than or equal to δ. This property reflects the geometric behavior of the space, and the fact that t, s, r are less than or equal to the respective distances to c_x, c_y, c_z ensures that the triangle is thin according to this definition.

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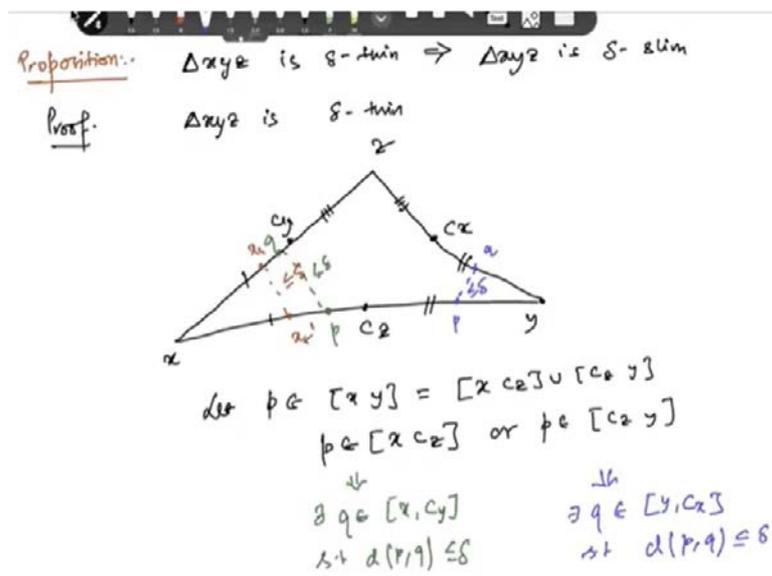
If a triangle Δ xyz is δ-thin, this implies that the triangle is also δ-slim. This follows directly from the definition of δ-thinness, and the proof is quite straightforward.

Let us assume that the triangle Δ xyz is δ-thin. Consider this triangle, and we can divide it into segments using the points c_x, c_y, and c_z. We know that the distance between x and c_y is equal to the distance between x and c_z, the distance between y and c_z is the same as the distance between y and c_x, and similarly, the distance between z and c_y is equal to the distance between z and c_x.

Now, for any points x_t and x_t' on the geodesics joining x to c_y and x to c_z, we know from the definition of thinness that the distance between x_t and x_t' is less than or equal to δ.

To prove that the triangle is δ-slim, let us take a point p that lies on the geodesic joining x and y (denoted by xy). The geodesic segment xy can be expressed as the union of two geodesics:

one joining x to c_z, and the other joining c_z to y. Thus, p must either lie on the geodesic joining x and c_z or on the geodesic joining c_z and y.

Suppose p lies on the geodesic joining x and c_z. Then, by the δ-thinness of the triangle, there exists a corresponding point q on the geodesic joining x and c_y, such that the distance between p and q is less than or equal to δ. This shows that for every point p on one side of the triangle, there exists a corresponding point q on the other side, with the distance between them bounded by δ.

Similarly, if p lies on the geodesic joining c_z and y, there will be a corresponding point q on the geodesic joining c_x and y, such that the distance between p and q is also less than or equal to δ. This follows directly from the property of δ-thinness.

Thus, for any point p on one side of the triangle, there is always a corresponding point q on the other side with a distance between them that is at most δ. This proves that if a triangle is δ-thin, it is also δ-slim.

Having established this, let us now proceed to another important proposition.

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Let δ $\geq$ 0 and consider (X, d) to be a geodesic metric space where all geodesic triangles are δ-slim. In such a scenario, there exists a non-negative number δ_1, which depends on δ, such that all triangles are also δ_1-thin. This means that given the assumption that the triangles are δ-slim, we can deduce the existence of δ_1, making the triangles δ_1-thin.

Let us now proceed with the proof.

Take a geodesic triangle Δ xyz in the space (X, d). From the hypothesis, we know that Δ xyz is δ-slim. Now, let's analyze the structure of this triangle by breaking it down, as we did before, into points c_x, c_y, and c_z, where:

- The distance between x and c_y is equal to the distance between x and c_z,
- The distance between y and c_z is equal to the distance between y and c_x,
- The distance between z and c_y is equal to the distance between z and c_x.

Since we have already assumed that this triangle is δ-slim, let us now focus on the point c_z. By the definition of δ-slimness, there exists a point c_z' that lies either on the geodesic between x and z, or on the geodesic between y and z, such that the distance between c_z and c_z' is less than or equal to δ.

Mathematically, this can be written as follows: there exists a point c_z' such that $c_z' \in (xz) \cup (yz)$ and $d(c_z, c_z') \leq \delta$. For the sake of simplicity, let us assume that c_z' lies on the geodesic joining x and z.

Next, note that this point c_z' may lie either on the geodesic joining x and c_y, or on the geodesic joining c_y and z. The important observation here is that the distances between corresponding points on these geodesics are bounded by δ, as dictated by the δ-slimness condition.

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By extending this argument to the entire triangle, we can conclude that the δ-slim property ensures the existence of δ_1, making all triangles δ_1-thin. This completes the proof.

The geodesic xz can be expressed as the union of the geodesic segments xc and cy. This means that the point c_z' lies within this union of the two segments, specifically within the geodesic segment xc and cy. Given that the distance between c_z' and c_y is less than or equal to δ, we can make some further deductions.

The distance between c_z' and c_y can be calculated as follows:

$$d\left(c_z', c_y\right) = d(xc) - d(x, c_z')$$

Since the distance $d(x, c_z')$ represents the length along the geodesic from x to c_z', we can substitute to find:

$$d\left(c_z', c_y\right) = d\left(x, c_y\right) - d(x, c_z')$$

Now, we know that $d(x, c_y)$ is equal to $d(x, c_z)$, thus allowing us to establish this equality. Applying the triangle inequality, we find:

$$d(x, z) \leq d(x, c_z') + d(c_z', c_z) - d(x, c_z')$$

Here, the terms $d(x, c_z')$ on both sides will cancel out, leading us to the conclusion that:

$$d(c_z, c_z') \leq \delta.$$

From this, we can say that the distance between c_y and c_z is less than or equal to:

$$d\left(c_y, c_z\right) \leq d\left(c_y, c_z'\right) + d(c_z', c_z) \leq 2\delta.$$

If we position c_z' along the segment xz, we can prove similarly that the distance between c_y and c_z will also be less than or equal to 2δ. I will leave this as an exercise for you.

Next, let's consider the point c_x. There exists a point c_x' that lies either on the geodesic joining x and z or on the geodesic joining x and y such that:

$$d(c_x, c_x') \leq \delta.$$

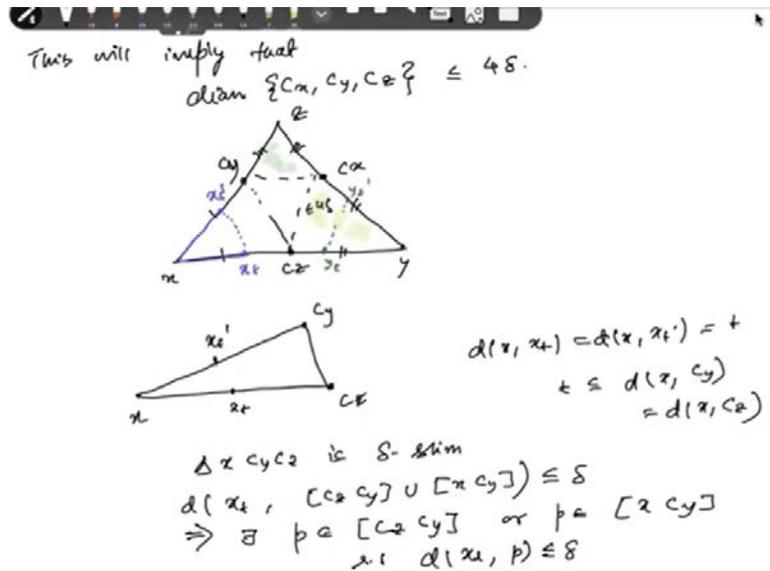Depending on the position of c_x', we can conclude either that:

$$d(c_x, c_y) \leq 2\delta.$$

or that:

$$d(c_x, c_z) \leq 2\delta.$$

Thus, we see that the relationships between these points are tightly bound by the condition of slimness, reinforcing the interconnectedness of the triangle's structure.

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This implies that the diameter of the set $\{c_x, c_y, c_z\}$ is less than or equal to 4δ. To recap what we've accomplished, we began with the triangle xyz and identified the points c_z, c_y, and c_x. We established that the distances among these points are equal, and consequently, the diameter of the set $\{c_x, c_y, c_z\}$ is indeed less than or equal to 4δ.

Now, our goal is to demonstrate that if we take the point x_t and another point $x_{t'}$ lying on the geodesic connecting x and c_y, the distances from x_t to $x_{t'}$ are equal. We aim to show that this distance is bounded in terms of δ. If we can establish this relationship, we will have accomplished our objective. We can then apply a similar argument to the other segment by considering points y_t and $y_{t'}$ in the corresponding manner.

Let us focus on the triangle xcy and the points x_t and $x_{t'}$ such that the distance from x to x_t is equal to the distance from x to $x_{t'}$. Let's denote this distance as t, where t is less than or equal to the distance between c_y and c_z, which is also the same as the distance between c_x and c_z. We

know that the triangle c_x c_y c_z is delta-slim, meaning that the distance from x_t to the union of c_y and the geodesic xc_y is less than or equal to δ.

This implies there exists a point p on the geodesic segment connecting c_z and c_y such that the distance from x_t to p is less than or equal to δ.

Now, let's consider the case where this point p belongs to the segment c_z c_y. Since we already established that the distance among the points c_x, c_y, c_z is less than or equal to 4δ, it follows that the distance between x_t and c_y will then be less than or equal to 5δ.

Thus, we have shown that if the distance from x_t to $x_{t'}$ is maintained in terms of δ, then we can extend this reasoning to our subsequent parts of the proof with similar findings.

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Now, suppose that the point p lies on the geodesic connecting x and c_y. In this scenario, p can belong to either the geodesic segment connecting x and t' or the segment connecting x and d' and c_y, which is clearly illustrated in the accompanying diagram.

Let's assume that p is situated on the geodesic connecting x and t'. In this case, the distance between p and t' can be expressed as the difference between the distance from x to p and the distance from x to t'. Thus, we have:

$$\text{distance}(p, t') = \text{distance}(x, p) - \text{distance}(x, t').$$

Since the distance from x to p is less than or equal to the sum of the distances from x to t and

from t to p, we can write:

$$\text{distance}(x, p) \leq \text{distance}(x, t) + \text{distance}(t, p) - \text{distance}(x, t').$$

Given that the distances from x_t to x_d' are equal, these two terms will cancel out, allowing us to simplify our expression. The distance from x_t to p is less than or equal to δ. Consequently, we can conclude that the distance between x_t and t' is at most 2δ when applying the triangle inequality, and thus we are done.

Similarly, if we place p on the other geodesic, we find that the distance between x_t and t' will also be less than or equal to 2δ)

Now, let's define δ_1 to be 6δ. Therefore, we have demonstrated that the triangle xyz is δ_1-thin. We have successfully proved the proposition. To conclude this lecture, I would like to make a final remark.

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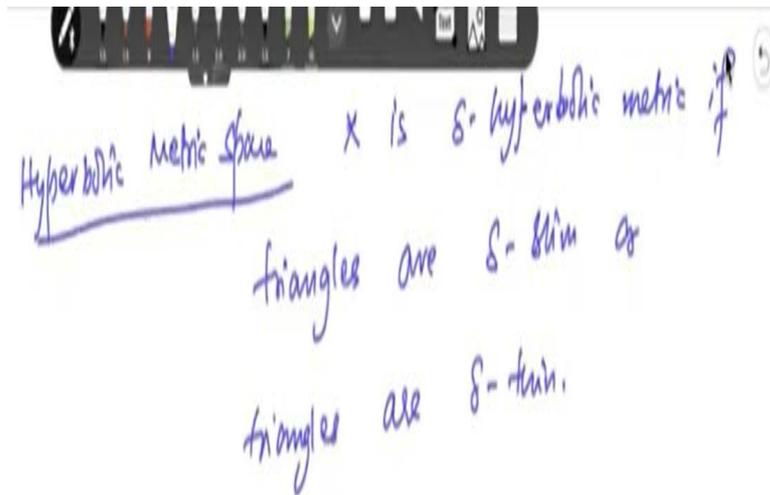
In the context of hyperbolic metric spaces, we define a space X as a δ-hyperbolic metric space if the triangles within that space are δ-slim. Alternatively, we can also characterize the space by stating that the triangles are δ-thin.

If we consider all triangles to be δ-thin, then we can affirmatively classify the space X as a δ-hyperbolic metric space. Conversely, if we only require the triangles to be δ-slim, this condition will also imply that X qualifies as a δ-hyperbolic metric space. With that, I will stop.