An Introduction to Hyperbolic Geometry

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Module No. 06

Lecture No. 30

Hyperbolic Geometry - Review

Hello! We have now reached an exciting stage in our course where we will dive into the fascinating realm of geometric group theory. To set the stage, let's first take a moment to review what we have accomplished thus far in our studies. Following that, I will elaborate on the intriguing topics that we will explore in this upcoming section on geometric group theory. So, let’s get started!

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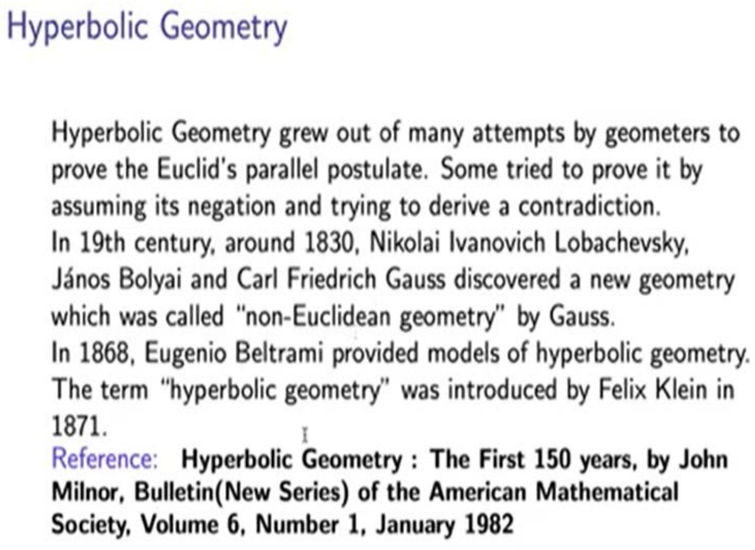

We began our journey with the foundational axioms of both equity and geometry. Hyperbolic geometry emerged from the efforts of geometers striving to prove the infamous fifth postulate, known as the parallel postulate. Some geometers attempted to establish its truth by assuming the negation of this postulate and seeking a contradiction.

In the nineteenth century, around the year 1830, pioneers such as Lvanovich, Lobachevsky, Bolyai, and Gauss made groundbreaking discoveries in this field. They uncovered a new form of geometry that was initially referred to as non-Euclidean geometry, a term first coined by

Gauss.

In 1868, Beltrami contributed by providing models of hyperbolic geometry, while the term "hyperbolic geometry" itself was introduced by Klein in 1871. For a more comprehensive history of hyperbolic geometry, I recommend referring to the insightful article by John Milnor.

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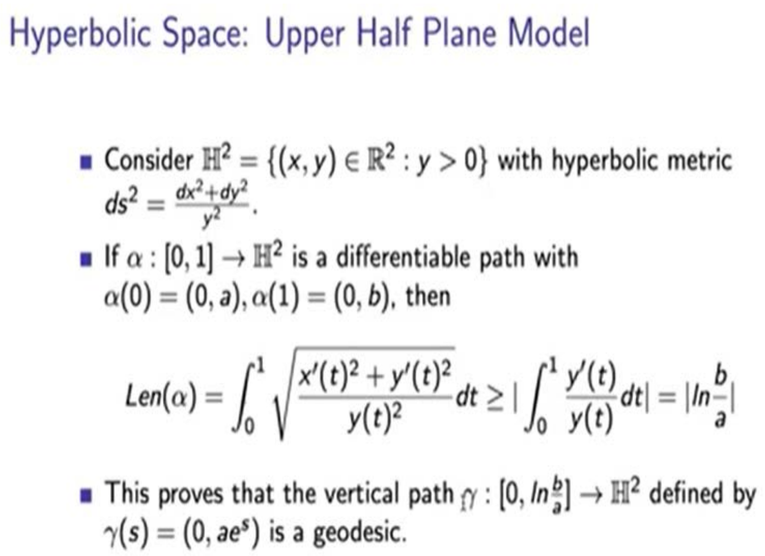

Let’s take a closer look at the upper half-plane model of hyperbolic space. We begin with this upper half-plane, where we have defined the hyperbolic metric as

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

With this metric, we can compute the length of any path, denoted as α. Suppose α(t) = (x(t), y(t)). The length of the path α can be expressed as

$$\text{Length}(\alpha) = \int_0^1 \sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2} \cdot \frac{1}{y(t)}\, dt,$$

where the interval [0, 1] serves as the domain of α. Importantly, we can assert that this length of α is greater than or equal to $|\log(b/a)|$, where α(0) = (0, a) and α(1) = (0, p).

Now, let’s consider a vertical path γ, which is defined from the interval 0 to $\log(p/a)$ in H^2. This path is given by

$$\gamma(s) = (0, ae^s),$$

where s serves as the length parameter. When we compute the length of γ, it turns out to be

$$|\log(b/a)|.$$

Since the length of α is greater than or equal to the length of γ, and the length of γ is exactly $|\log(b/a)|$, we conclude that γ is indeed a geodesic. Thus, we see that the path γ represents the shortest distance between the points in this hyperbolic space.

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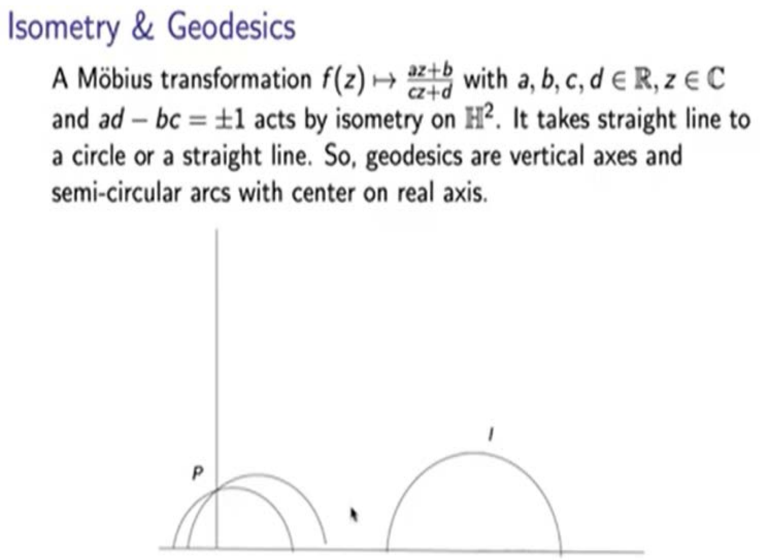

The vertical paths are indeed geodesics. Now, let's consider a Möbius transformation defined as

$$\phi(z) = \frac{az+b}{cz+d},$$

where a, b, c, and d are real numbers. If we impose the condition that $ad - bc = \pm 1$, this Möbius transformation acts as an isometry on the upper half-plane. One remarkable property of Möbius transformations is that they map any circle or straight line to another circle or straight line.

Consequently, we observe that the geodesics in this model take the form of vertical lines and semicircular arcs with their centers located on the real axis. This leads us to a crucial observation: the fifth postulate of Euclidean geometry is indeed violated in this context.

Specifically, given a point p and a geodesic l in the hyperbolic plane, if p is not on the line l, then there exist infinitely many lines passing through the point p that do not intersect l.

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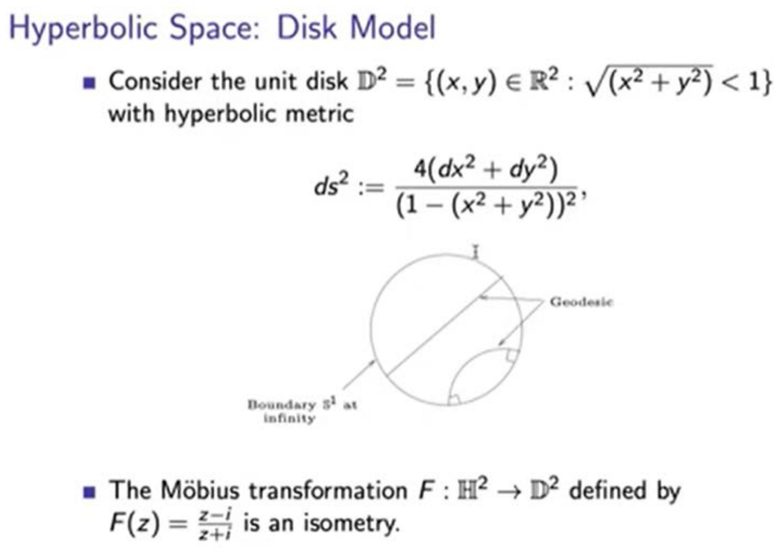

There is another fascinating model known as the Poincaré disk model. Let's consider this unit disk and define a metric given by

$$ds^2 = \frac{4\,dx^2 + dy^2}{(1 - x^2 + y^2)^2}.$$

In this context, we have a mapping from the upper half-plane H^2 to the disk D^2, specifically defined as

$$f(z) = \frac{z - i}{z + i}.$$

Here, H^2 is equipped with its own hyperbolic metric, and D^2 has a corresponding metric as well. Notably, this map f is an isometry, meaning it preserves distances.

In the Poincaré disk model, geodesics take the form of diameters and circular arcs that intersect the unit circle at right angles. It’s important to recognize that the unit circle in this model is considered to be at infinity. Similarly, in the upper half-plane model, the line $R \cup \{\infty\}$ also represents a boundary at infinity, which is not located at any finite distance from points in the upper half-plane.

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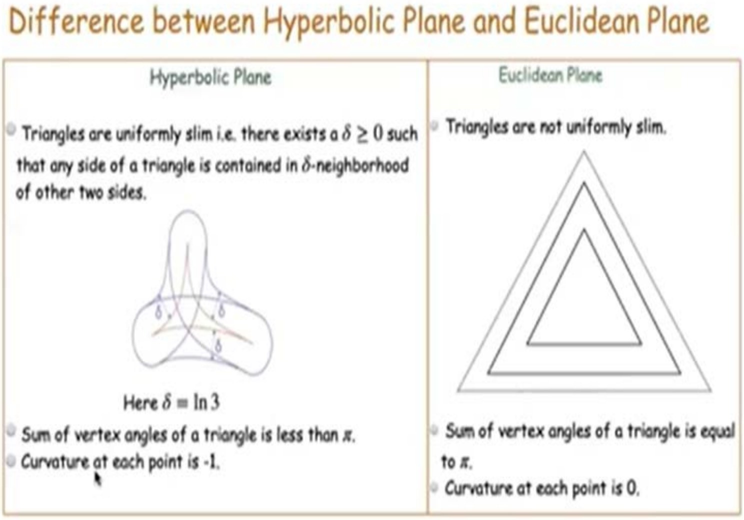

Let's examine the fundamental differences between the hyperbolic plane and the Euclidean plane. In hyperbolic geometry, triangles are uniformly slim. This means that there exists a non-negative number, δ, such that any side of a triangle is contained within a δ-neighborhood of the union of the other two sides. For instance, in the upper half-plane or the Poincaré disk model, this δ can be expressed as log 3.

In contrast, triangles in the Euclidean plane are not slim; instead, they are what we might describe as "fat." By "fat," I mean that the triangles can grow larger and larger without bound as they expand outward in the Euclidean plane.

Moreover, in hyperbolic geometry, the sum of the vertex angles of a triangle is always less than π radians. In the Euclidean plane, however, the sum of the vertex angles of a triangle is precisely equal to π radians. This difference can be attributed to the curvature of the surfaces: in hyperbolic geometry, the curvature at each point of the hyperbolic plane is -1, indicating a saddle-like shape, whereas in the Euclidean plane, the curvature at every point is 0, reflecting a flat surface.

We have explored the isometries of the hyperbolic plane, particularly focusing on the upper half-plane model. In this context, the group of orientation-preserving isometries is isomorphic to $\mathrm{PSL}_2(R)$. We also delved into Fuchsian groups, which are discrete subgroups G of $\mathrm{PSL}_2(R)$. The term "Fuchsian" was first introduced by Henri Poincaré, who was inspired by the work of Fuchs in the 1880s, leading to the naming of this mathematical concept.

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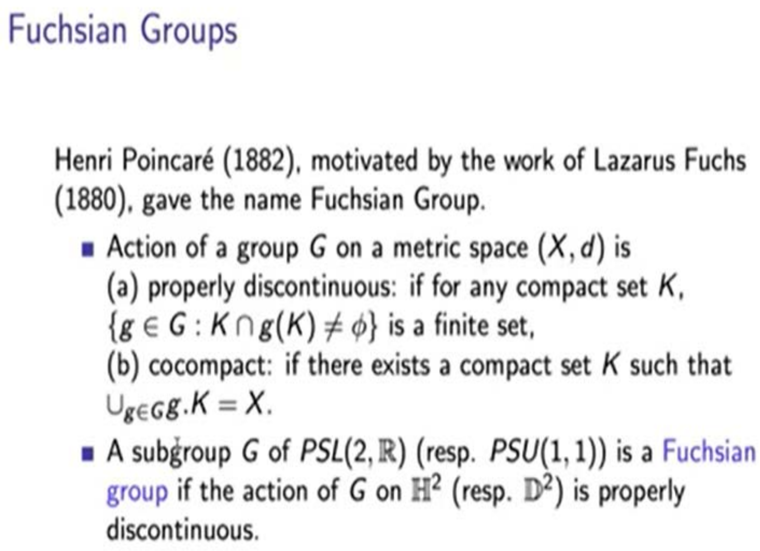

Now, let's recall the definition of a properly discontinuous action. An action of a group is said to be properly discontinuous if, for any compact set K, the intersection of the set $\{ g \in G \mid K \cap gK \neq \emptyset\}$ is finite. Additionally, we say that the action is cocompact if there exists a compact set K such that the union of gK covers the entire space X. Here, gK denotes the action of an element g from the group G on the compact set K.

A subgroup G of $\mathrm{PSL}_2(R)$ is classified as a Fuchsian group if its action on the upper half-plane is properly discontinuous. Equivalently, any orbit under this action will yield a discrete subset of the upper half-plane, a concept we have already discussed.

As for examples of Fuchsian groups, consider $\mathrm{PSL}_2(Z)$; it is indeed a discrete subgroup of $\mathrm{PSL}_2(R)$. We have also examined elementary and non-elementary Fuchsian groups, demonstrating that surface groups fall into the category of non-elementary Fuchsian groups.

Let's revisit an important result we've established: if we take any compact surface without boundary, and if the genus of that surface is greater than or equal to 2, then it admits a hyperbolic metric. For example, consider S^2 as a closed surface of genus 2. This surface can be obtained as a quotient space of a specific geodesic octagon, as illustrated in the accompanying diagram.

In this diagram, we identify the side v_1v_4 with the side v_2v_3 through an isometry. There exists an isometry A that maps the side v_1v_4 to v_2v_3. We proceed similarly for the other edges, creating a series of side pairings for this geodesic octagon. The result of these identifications is the genus

2 surface.

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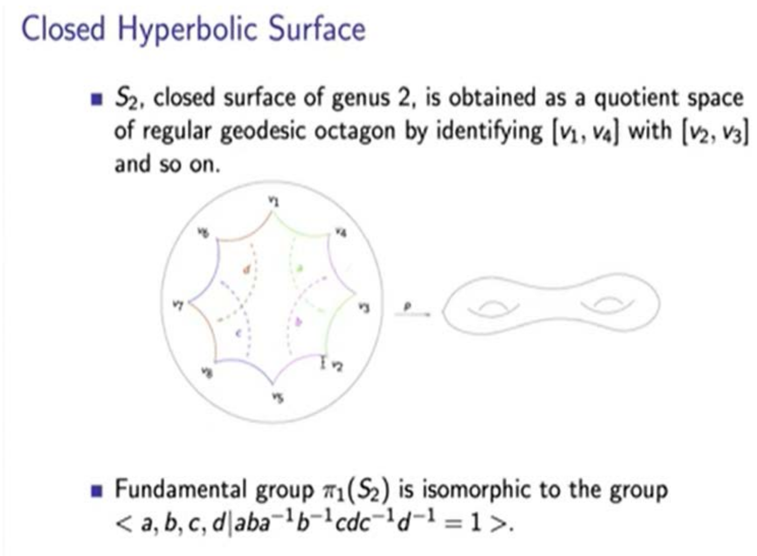

Moreover, we have observed that the fundamental group of this genus 2 surface is generated by the elements a, b, c, and d. This group is a one-relator group, and the relation can be expressed as the product of the commutators equaling 1. Specifically, this means that the relation can be written as:

$$aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1.$$

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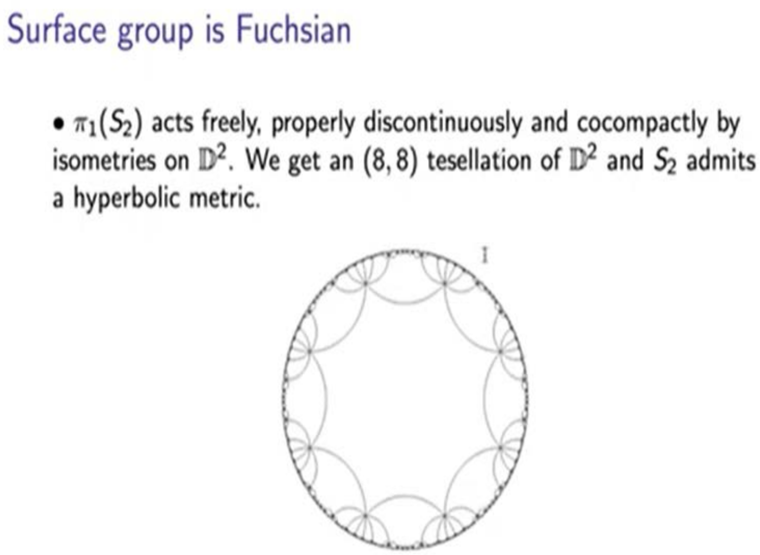

Thus, we have established that this fundamental group is indeed a subgroup of PSL_1 or $\mathrm{PSL}_2(R)$, which in turn implies that it is a Fuchsian group.

We have observed that the fundamental group of S^2 acts freely and properly discontinuously, and importantly, this action is also co-compact. The fundamental group $\pi_1(S^2)$ acts by isometries on the upper half-plane model of hyperbolic space. Consequently, this action is free, properly discontinuous, and co-compact.

As a result, we obtain an 8,8 tessellation of the unit disk. Through our exploration of covering space theory, we have established that this surface S^2 indeed admits a hyperbolic metric. This connection between the fundamental group and the geometry of the surface reinforces our understanding of hyperbolic geometry and its underlying structures.

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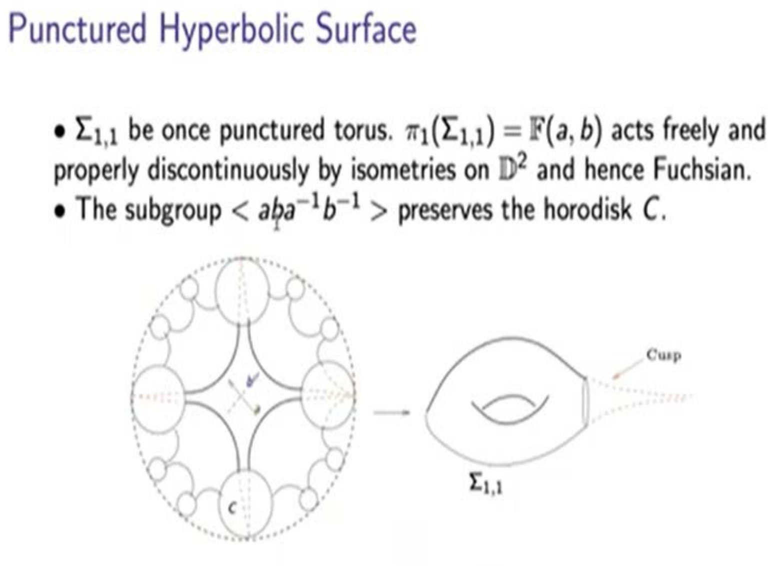

Similarly, when we consider the punctured surface Σ_1, which consists of a surface with one puncture and one genus, essentially forming a punctured torus, we find that the fundamental group $\pi_1(\Sigma_1)$ is a free group generated by two elements, denoted as a and b. We have also observed that this group acts freely and properly discontinuously by isometries on a particular disk, which leads to the notion of quotient spaces.

Moreover, you can verify that the commutator $aba^{-1}b^{-1}$ generates a cyclic subgroup that preserves the horodisk C illustrated in this context. This relationship is quite significant, and I will now proceed to the next slide for further elaboration.

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Geometric Group Theory

Geometric Group Theory is the branch of mathematics that studies finitely generated groups via its action on geometric objects like Euclidean space, Hyperbolic Space, Trees etc.
The subject was popularized by the foundational work of Mikhael Gromov (in 1987) on Hyperbolic Groups, these are groups whose Cayley Graphs are "hyperbolic metric spaces".
Hyperbolic Metric Spaces are central geometric objects in geometric group theory which captures the negative curvature of a metric space.

Now, let us transition to the segment on geometric group theory. This branch of mathematics focuses on the study of finitely generated groups and their actions on geometric objects, such as Euclidean space, hyperbolic space, and trees. For instance, consider Z^2; it possesses a natural action on R^2 equipped with the Euclidean metric. In this case, Z^2 acts on R^2 through translations, which exemplifies a straightforward action.

Similarly, when we examine the fundamental group of a genus two surface or any surface of higher genus, we find that it acts elegantly on the hyperbolic plane. Additionally, if we consider a free group with two generators or an infinite cyclic group, we can observe their action on the real line R. Furthermore, a free group with two generators will act on a regular tree where each vertex has a degree of four. These concepts will be explored in detail throughout this course.

Geometric group theory gained significant attention through the work of Gromov in 1987, who introduced the notion of hyperbolic groups. But what exactly are hyperbolic groups? They are groups whose Cayley graphs exhibit the properties of hyperbolic metric spaces. Throughout this course, we will engage with these hyperbolic groups and delve into the associated hyperbolic metric spaces, which represent fundamental geometric objects in geometric group theory and encapsulate the essence of negative curvature within a metric space.

Let us now quickly review the definitions and foundational concepts related to these topics.

We have previously defined slim triangles in the context of hyperbolic geometry, and now we can extend this concept to any metric space. To do so, let's start with a non-negative number, denoted as δ. A geodesic triangle is considered to be δ-slim if each side of the triangle is contained within a δ-neighborhood of the union of the other two sides. This δ-slim property for triangles has been well illustrated in the upper half-plane equipped with hyperbolic metrics.

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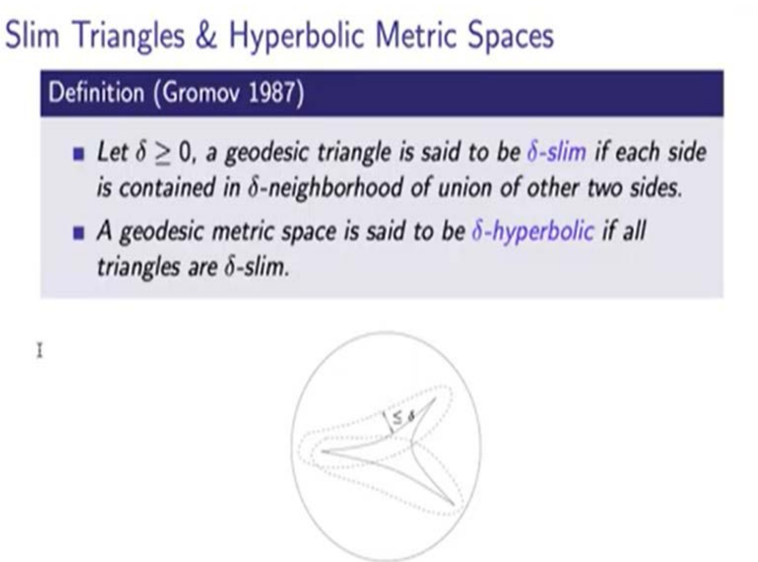

Now, let's clarify what it means for a geodesic metric space to be δ-hyperbolic. A geodesic metric space is deemed δ-hyperbolic if all triangles within that space are δ-slim.

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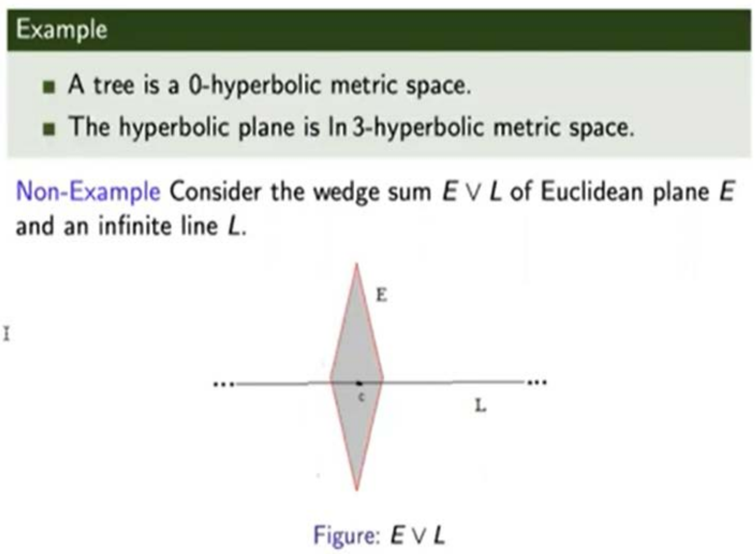

This property of hyperbolicity captures the essence of the geometric structure of the space and provides a framework for understanding its intrinsic characteristics.

We have previously established that the hyperbolic plane is a log 3-hyperbolic metric space, meaning that all the triangles within this space are log 3-slim. Now, let's discuss the concept of a tree. A tree is defined as a graph that contains no cycles, and interestingly, it is a 0-hyperbolic metric space. This implies that if we consider any triangle within a tree, it will be a triangle with zero hyperbolicity. Therefore, here, δ is equal to 0, affirming that a tree is indeed a 0-hyperbolic metric space.

Now, let's consider a non-example to illustrate this further. The Euclidean plane, for instance, is clearly not a hyperbolic metric space since the triangles formed there are not slim. Moreover, you can construct various spaces from the Euclidean plane that also fail to be hyperbolic. For example, let's take E to be the Euclidean plane and L to be an infinite line. If we consider the wedge sum of E and L, these two intersect at only a single point, denoted as C. Despite the Euclidean plane being isometrically embedded in this space, it results in a space that is not hyperbolic.

However, it is noteworthy that the subset L is indeed a hyperbolic metric space, as it can be considered a tree. Thus, while L qualifies as a hyperbolic metric space, the entirety of the constructed space does not share this property.

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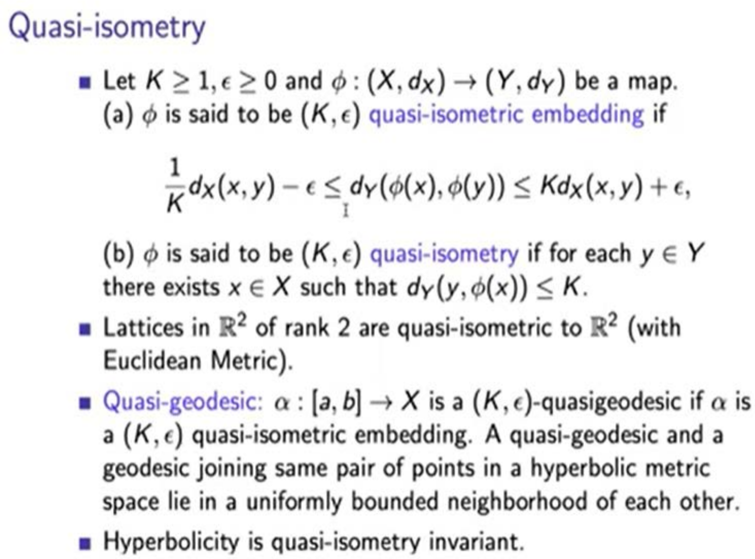

In our study, we will also explore a specific type of mapping known as quasi-isometric maps.

A map φ from a space X to a space Y is considered to be kϵ-quasi-isometric if it satisfies the following inequality:

$$\frac{1}{K} \cdot d(x,y) - \epsilon \leq d\big(\varphi(x), \varphi(y)\big) \leq k \cdot d(x,y) + \epsilon,$$

If this inequality holds, we refer to φ as a quasi-isometric embedding. Furthermore, φ is termed quasi-isometric if, for every point y in the codomain, there exists a point x in the domain such that the distance between y and φ(x) is less than or equal to K.

One can demonstrate that lattices in R^2 of rank 2 are quasi-isometric to R^2, where R^2 is considered with the Euclidean metric. In this course, we will also examine quasi-geodesics.

So, what exactly are quasi-geodesics? If we take a closed interval [a, b] and define a map α from this interval to a metric space X, then α is classified as a kϵ-quasi-geodesic if it serves as a kϵ-quasi-isometric embedding. It's important to note that a quasi-geodesic may not necessarily be continuous in terms of the metric.

We will also prove the stability property of quasi-geodesics and geodesics. This property states that any geodesic connecting the same pair of points in a hyperbolic metric space will lie within a uniformly bounded neighborhood of each other. This phenomenon is referred to as the stability of quasi-geodesics.

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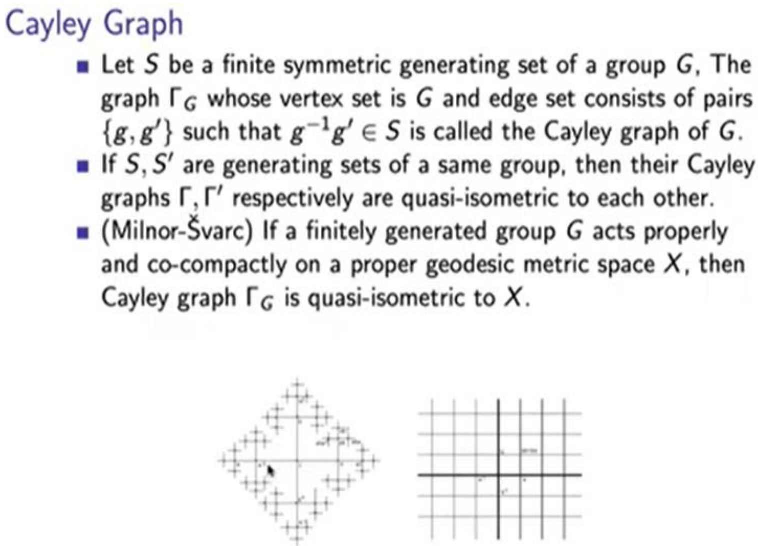

Utilizing this property, we can conclude that hyperbolicity is quasi-isometric invariant.

Specifically, if we have a quasi-isometric map from X to Y and know that X is a hyperbolic metric space, then Y will also be a hyperbolic metric space.

In this section, we will delve into the concept of Cayley graphs, which correspond to curves in mathematical spaces. Let's consider a finitely generated group G with a finite symmetric generating set S. The Cayley graph, denoted as Γ(G), is constructed with the vertex set consisting of the elements of G. The edge set consists of pairs (g, g') such that the inverse of g multiplied by g' is an element of S. This construction forms what we refer to as the Cayley graph of G.

As we explore this in detail, we will also encounter cases where a group has two generating sets, say S and S'. In such scenarios, we can demonstrate that the Cayley graphs Γ and Γ' are quasi-isometric to each other. This fascinating result is a well-celebrated consequence of the work by John Milnor.

Now, let’s consider a finitely generated group G that acts properly and cocompactly on a proper geodesic metric space X. In this case, it can be shown that the Cayley graph of the group G is quasi-isometric to the space X.

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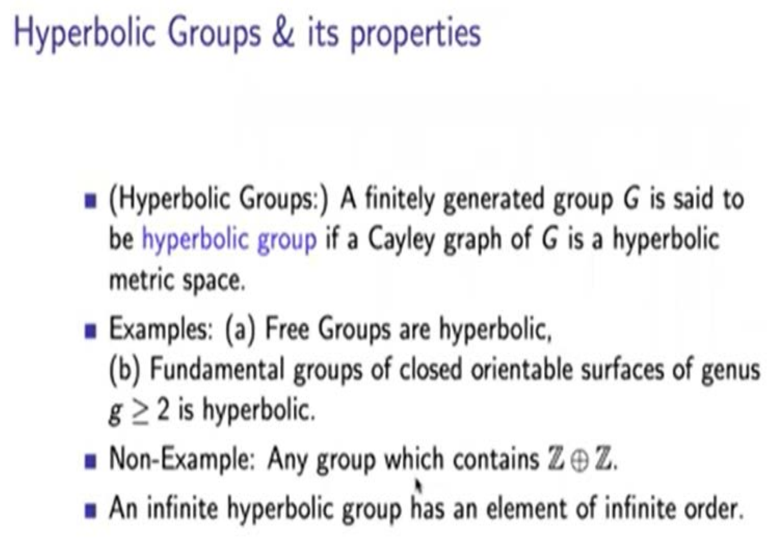

Here are a couple of examples to illustrate the concept of Cayley graphs: If we take the free group with two generators, a and b, we find that the corresponding Cayley graph forms an infinite tree where the degree of each vertex equals 4. Additionally, for the group Z^2, its Cayley graph corresponds to a square lattice of rank 2, showcasing the intrinsic structure of this group.

Now, let's define what constitutes a hyperbolic group. A finitely generated group G is classified as hyperbolic if its Cayley graph is a hyperbolic metric space. In other words, if we start with a group G that is finitely generated, and its Cayley graph exhibits hyperbolic properties, then we refer to G as a hyperbolic group.

For instance, consider free groups. As mentioned earlier, the Cayley graphs of free groups take the form of trees, and trees themselves are hyperbolic metric spaces. Consequently, we can confidently state that free groups are indeed hyperbolic.

We've also established that the fundamental groups of closed orientable surfaces with genus $g \geq 2$ act properly discontinuously and cocompactly on a certain disk. Utilizing the lemma derived from Milnor's work, we can conclude that these fundamental groups are quasi-isometric to this particular disk. Thus, it follows that the fundamental groups of closed orientable surfaces with genus greater than or equal to 2 are hyperbolic.

Now, let's discuss non-examples. Any group that contains Z^2 or $Z \times Z$ will not be hyperbolic. As we continue our exploration, we will examine the properties of hyperbolic groups. One notable property is that for any infinite hyperbolic group, one can prove the existence of an element with infinite order.

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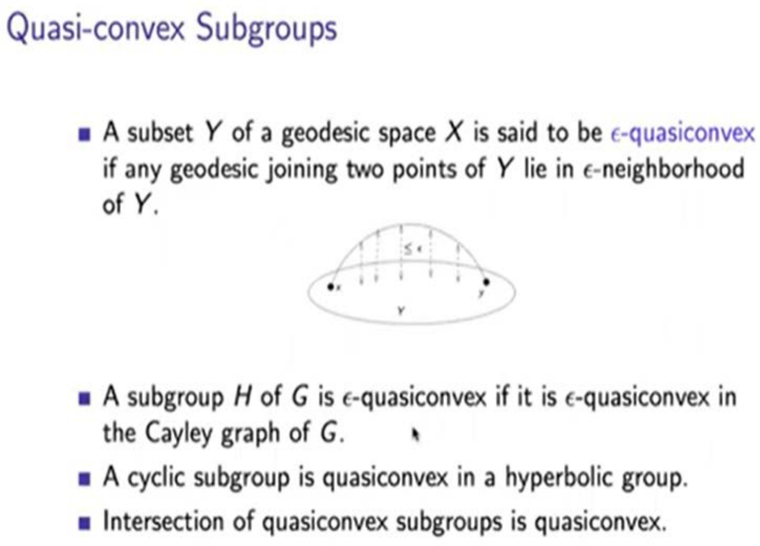

We will now explore the concept of quasi-convexity within hyperbolic groups. A subset Y of a geodesic space is defined as ϵ-quasi-convex, where ϵ is a non-negative number that has already been predetermined. Specifically, the subset Y of the space X is deemed ϵ-quasi-convex if any geodesic connecting two points within Y lies entirely within the ϵ-neighborhood of Y. This means that any geodesic segment joining two points in Y will be situated in X, and it will fall within the ϵ-neighborhood of Y if Y satisfies the condition of being ϵ-quasi-convex.

Additionally, a subgroup H of G is classified as ϵ-quasi-convex if it is ϵ-quasi-convex in the Cayley graph of G.

Throughout our study, we will examine various properties of quasi-convex groups. Notably, we will demonstrate that a cyclic subgroup is always quasi-convex within a hyperbolic group. Furthermore, we will prove that the intersection of two quasi-convex subsets remains quasi-convex.

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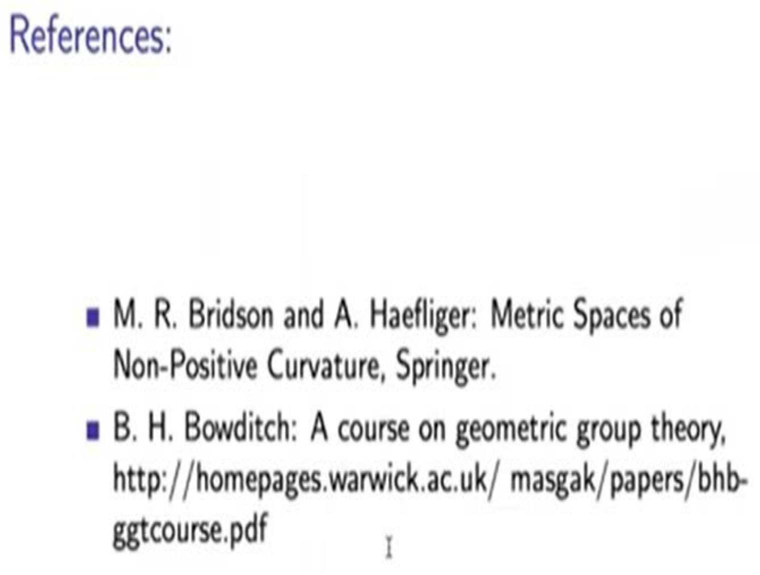

For our reference, we will refer to two important resources. The first is the book by Bridson and Haefliger, titled Metric Spaces of Non-Positive Curvature. This work provides valuable insights and foundational knowledge in the field. Additionally, we will also consult the notes by Bowditch, titled A Course on Geometric Group Theory. These resources will serve as essential guides throughout our study. I will stop here.