

# An Introduction to Hyperbolic Geometry

Prof. Abhijit Pal

Department of Mathematics and Statistics

Indian Institute of Technology - Kanpur

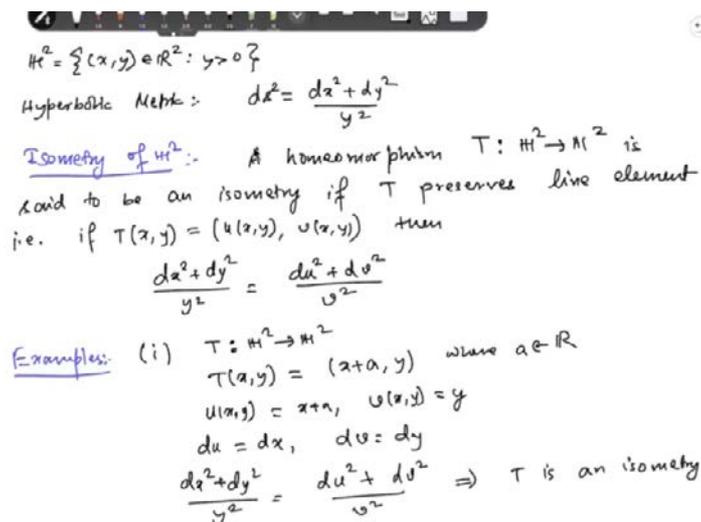
Module - 1

Lecture - 3

## An Introduction to Hyperbolic Geometry

Hello everyone, and welcome to this lecture on Hyperbolic Geometry. Today, we will delve into the fascinating topic of isometries within the upper half-plane model of the hyperbolic plane. Specifically, we will be proving that the group  $PSL(2, R)$  is indeed a subgroup of the isometry group for this upper half-plane. Let's begin exploring this important relationship and understand how this group preserves the hyperbolic structure.

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$\mathbb{H}^2 = \{ (x, y) \in \mathbb{R}^2 : y > 0 \}$   
Hyperbolic Metric:  $ds^2 = \frac{dx^2 + dy^2}{y^2}$   
Isometry of  $\mathbb{H}^2$ : A homeomorphism  $T: \mathbb{H}^2 \rightarrow \mathbb{H}^2$  is said to be an isometry if  $T$  preserves line element  
i.e. if  $T(x, y) = (u(x, y), v(x, y))$  then  
$$\frac{dx^2 + dy^2}{y^2} = \frac{du^2 + dv^2}{v^2}$$
  
Examples: (i)  $T: \mathbb{H}^2 \rightarrow \mathbb{H}^2$  where  $a \in \mathbb{R}$   
 $T(x, y) = (x+a, y)$   
 $u(x, y) = x+a, v(x, y) = y$   
 $du = dx, dv = dy$   
$$\frac{dx^2 + dy^2}{y^2} = \frac{du^2 + dv^2}{v^2} \Rightarrow T \text{ is an isometry}$$

Let's begin by briefly recalling what we covered in the last class. Here, we are working with the upper half-plane, represented as the ordered pair  $(x, y)$ , where both  $x$  and  $y$  are real numbers, and  $y > 0$ . We introduced a metric on this upper half-plane, which we called the hyperbolic metric. Specifically, this metric is given by the square of the line element:

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

This is referred to as the hyperbolic metric or, alternatively, the Poincaré metric. In this lecture, we will demonstrate that the group  $SL(2, R)$ , or more precisely  $PSL(2, R)$ , acts on the upper half-plane with the hyperbolic metric, preserving isometries.

To proceed, let me first define what we mean by an isometry. An isometry is a homeomorphism  $T$  from the upper half-plane  $H^2$  to itself, which preserves the line element. In other words,  $T$  is an isometry if it maintains the form of the metric under transformation.

So, we have a map  $T: H^2 \rightarrow H^2$ , and we express it as  $T(x, y) = (u(x, y), v(x, y))$ . For  $T$  to be an isometry, it must satisfy the following condition:

$$\frac{dx^2 + dy^2}{y^2} = \frac{du^2 + dv^2}{v^2}$$

If this condition holds, then we can say  $T$  is indeed an isometry.

Now, let's look at some examples of isometries. Consider the map  $T: H^2 \rightarrow H^2$  defined by  $T(x, y) = (x + a, y)$ , where  $a$  is a real number. In this case, the map shifts  $x$  by a constant  $a$ , while leaving  $y$  unchanged. So, we have  $u(x, y) = x + a$  and  $v(x, y) = y$ .

For this transformation, the differentials  $du = dx$  and  $dv = dy$ , which shows that the formula for the square of the line element is preserved. Hence, this transformation  $T$  preserves the hyperbolic metric, and we conclude that  $T$  is an isometry.

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(i)  $T_\lambda(x, y) = (\lambda x, \lambda y)$ ,  $\lambda > 0$   
 $u(x, y) = \lambda x$ ,  $v(x, y) = \lambda y$   
 $du = \lambda dx$ ,  $dv = \lambda dy$   
 $\frac{du^2 + dv^2}{v^2} = \frac{\lambda^2(dx^2 + dy^2)}{\lambda^2 y^2} = \frac{dx^2 + dy^2}{y^2}$   
 $T_\lambda$  is an isometry of  $H^2$ .

(ii)  $T_\lambda(x, y) = (\lambda x, \lambda y)$

(i)  $(x, y) \rightarrow (x+a, y)$   


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 $T(z) = z + a$ ,  $z = x + iy$   
 $\downarrow$   
 $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$

(ii)  $T_\lambda(z) = \lambda z = \frac{\sqrt{a} z}{\sqrt{a}}$   
 $\downarrow$   
 $z = x + iy$   
 $\begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{a} \end{pmatrix}$

This provides an example of how certain transformations preserve the hyperbolic structure, and we'll explore more such transformations as we proceed.

Let's move on to another example. Consider the transformation  $T(x, y)$  defined as  $T(x, y) = (\lambda x, \lambda y)$ , where  $\lambda$  is a positive real number. In this case, the functions  $u(x, y)$  and  $v(x, y)$  become  $u(x, y) = \lambda x$  and  $v(x, y) = \lambda y$ . Therefore, the differentials are  $du = \lambda dx$  and  $dv = \lambda dy$ .

Now, if we substitute this into the metric formula, the numerator becomes  $\lambda^2(dx^2 + dy^2)$ , and the denominator becomes  $\lambda^2 y^2$ . This simplifies to the original hyperbolic metric  $\frac{dx^2+dy^2}{y^2}$ , meaning the transformation  $T$  preserves the metric. Hence, this transformation  $T$  is indeed an isometry of the upper half-plane.

Now, let's compare the geometric effects of the two examples we've discussed. In the first example, we had a horizontal translation on the upper half-plane, where  $T(x, y) = (x + a, y)$ . So, if we take a point  $(x, y)$  and apply this transformation, the point simply moves horizontally along the  $x$ -axis without changing the  $y$ -coordinate. Thus, a horizontal translation is an isometry of the upper half-plane.

In the second example, we defined  $T(x, y) = (\lambda x, \lambda y)$ , which represents a scaling transformation. If we take a point  $(x, y)$  and apply this transformation, the new coordinates become  $(\lambda x, \lambda y)$ , essentially scaling the point away from or toward the origin, depending on  $\lambda$ . So, this scaling by  $\lambda$  is also an isometry.

In terms of complex number notation, we can express these transformations more concisely. For the first map, we can write it as  $T(z) = z + a$ , where  $z = x + iy$ . For the second map, we can write it as  $T(\lambda z) = \lambda z$ .

Corresponding to the first map,  $T(z) = z + a$ , we can associate the matrix  $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ , which has a determinant of 1, confirming its role in preserving the structure of the upper half-plane.

Similarly, for the second map,  $T(\lambda z) = \lambda z$ , we can associate the matrix  $\begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix}$ , which also has a determinant of 1.

These observations give us a strong indication that the group  $SL(2, R)$ , or more specifically  $PSL(2, R)$ , acts on the upper half-plane by isometries. So now, we are ready to define the action

of  $SL(2, R)$  on the upper half-plane. Let's proceed with that definition.

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Action of  $SL(2, R)$  on  $H^2$

$$SL(2, R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in R, ad - bc = 1 \right\}$$

$$SL(2, R) \times H^2 \rightarrow H^2$$

$$\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) \mapsto \frac{az + b}{cz + d}$$

if  $z \in H^2$  then  $\frac{az + b}{cz + d} \in H^2$  where  $a, b, c, d \in R$   
 $ad - bc = 1$

$$\text{Im} \left( \frac{az + b}{cz + d} \right) = \frac{\text{Im}(z)}{|cz + d|^2} > 0$$

$$\Rightarrow \frac{az + b}{cz + d} \in H^2$$

Also,  $T(z) = \frac{az + b}{cz + d}$ ,  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, R)$   
 is a homeomorphism.

Let's discuss the action of  $SL(2, R)$  on the upper half-plane. First, let's clarify what the group  $SL(2, R)$  is. This group consists of all  $2 \times 2$  matrices of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where  $a, b, c, d$  are real numbers, and the determinant of this matrix is equal to 1.

Now, we will define the action of  $SL(2, R)$  on the upper half-plane  $H^2$ . For a point  $z$  in  $H^2$ , which we can think of as a complex number, the action is defined as follows:

$$T(z) = \frac{az + b}{cz + d}$$

This definition encapsulates how the group acts on points in the upper half-plane.

It is straightforward to verify that if  $z$  belongs to  $H^2$ , then  $T(z) = \frac{az + b}{cz + d}$  will also belong to  $H^2$ .

Specifically, if  $z$  is in  $H^2$ , the imaginary part of  $T(z)$  can be expressed as follows:

$$\text{Im}(T(z)) = \frac{\text{Im}(az + b)}{|cz + d|^2}$$

This is guaranteed to be greater than zero, confirming that  $T(z)$  indeed lies within the upper

half-plane.

Additionally, we can check that the transformation  $T$  is a homeomorphism. This means that  $T$  establishes a continuous one-to-one mapping from  $H^2$  to itself. If you're familiar with complex analysis, you'll recognize that the action of  $SL(2, R)$  on the upper half-plane is realized through what is known as Möbius transformations.

Thus, we have:

$$T(z) = \frac{az + b}{cz + d}$$

And as we know, Möbius transformations are always homeomorphisms. Now, what we will do next is to define another aspect related to this action.

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$PSL(2, R) := \frac{SL(2, R)}{\{\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}}$

**Theorem:**  $PSL(2, R)$  is isomorphic to a subgroup of  $Isom(H^2)$

$(*) Isom(H^2) = \{ f: H^2 \rightarrow H^2 \mid f \text{ is an isometry} \}$   
 $Isom(H^2)$  forms a group under the composition of maps.

**Proof:**  $\phi: PSL(2, R) \rightarrow Homeo(H^2)$   
 $\phi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \stackrel{\text{def.}}{=} \left( z \mapsto \frac{az+b}{cz+d} \right)$

We will prove that any Möbius transformation  $T(z) = \frac{az+b}{cz+d}$ ,  $ad-bc=1$ ,  $a, b, c, d \in R$ , is an isometry of  $H^2$ .

Let  $\gamma: [a, b] \rightarrow H^2$  be a piecewise differentiable path in  $H^2$ . **Claim:**  $l(T(\gamma)) = l(\gamma)$

Let's establish that  $PSL(2, R)$  is isomorphic to a subgroup of isometries of the upper half-plane. What does this entail? Essentially, we are considering the set of all isometries of the upper half-plane, which consists of mappings from  $H^2$  to  $H^2$  that form a group under the operation of composition.

Now, let's delve into the proof of this concept. We will start by defining a map, which we will denote as  $\phi$ . This isometry of the upper half-plane is defined as follows:

$$\phi(z) = \frac{az + b}{cz + d}$$

Our goal is to demonstrate that this map is indeed an isometry of the upper half-plane. In fact, we can establish a map from  $\text{PSL}(2, R)$  to the set of all self-homeomorphisms of the upper half-plane. This means that the set of homeomorphisms from  $H^2$  to  $H^2$  also forms a group.

To prove our assertion, we first focus on the general case of Möbius transformations. Consider a transformation given by

$$T(z) = \frac{az + b}{cz + d}$$

where  $a, b, c, d$  are real numbers, and the condition  $ad - bc = 1$  holds. We need to show that this transformation is an isometry of the upper half-plane.

To accomplish this, let's take a piecewise differentiable path  $\gamma$  in the upper half-plane. Our claim is that the length of the transformed path  $T(\gamma)$  is equal to the length of the original path  $\gamma$ .

Now, let's proceed to prove this assertion.

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$\text{let } \gamma(t) = x(t) + iy(t), t \in [a, b]$   
 $\& \text{ let } T(z) = u(x, y) + iv(x, y), z = x + iy$   

$$L(T(\gamma)) = \int_a^b \sqrt{\left(\frac{du}{dt}\right)^2 + \left(\frac{dv}{dt}\right)^2} dt \quad (1)$$
  

$$\sqrt{\left(\frac{du}{dt}\right)^2 + \left(\frac{dv}{dt}\right)^2} = \left| \frac{d}{dt} T(\gamma(t)) \right| = |T'(\gamma(t)) \cdot \gamma'(t)|$$

$$= |T'(\gamma(t))| |\gamma'(t)| \quad (2)$$
  

$$T(z) = \frac{az + b}{cz + d}$$

$$T'(z) = \frac{1}{(cz + d)^2} \quad \text{check}$$

$$\text{Im } T(z) = \frac{\text{Im } z}{|cz + d|^2}$$

$$|T'(z)| = \frac{\text{Im } T(z)}{\text{Im } z} \Rightarrow |T'(\gamma(t))| = \frac{v(x(t), y(t))}{y(t)} \quad (3)$$

We start by considering the path  $\gamma(t)$ , where  $\gamma(t) = x(t) + iy(t)$ , and  $t$  lies in the interval  $[a, b]$ . Here,  $x(t)$  and  $y(t)$  represent the real and imaginary parts of the complex number  $z = x + iy$ .

Now, consider the following integral:

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \frac{dt}{y(t)}$$

We'll refer to this equation as Equation (1). This expression represents the length of the path  $\gamma(t)$  in the hyperbolic metric.

Next, we introduce the transformation  $T$  and its derivative. The norm of the derivative of  $T$  along the path  $\gamma(t)$  can be written as:

$$|T'(\gamma(t))| \cdot |\gamma'(t)|$$

We refer to this equation as Equation (2). This expression indicates that the length of the transformed path is related to the derivative of the transformation and the derivative of the original path.

Now, let's evaluate  $T'(z)$  for the Möbius transformation  $T(z) = \frac{az+b}{cz+d}$ . We find that:

$$T'(z) = \frac{1}{(cz + d)^2}$$

To proceed, we also need to express the imaginary part of  $T(z)$ . This is given by:

$$\text{Im}(T(z)) = \frac{\text{Im}(z)}{|cz + d|^2}$$

From this, we deduce that:

$$|T'(z)| = \frac{\text{Im}(T(z))}{\text{Im}(z)}$$

Now, applying this to our path  $\gamma(t)$ , we find:

$$|T'(\gamma(t))| = \frac{y(t)}{|cz(t) + d|}$$

Let's refer to this as Equation (3).

By combining these three equations, namely Equation (1), Equation (2), and Equation (3), we

can conclude that the length of the path under the Möbius transformation is preserved, demonstrating that  $T(z)$  is indeed an isometry.

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From (1), (2), (3)

$$\begin{aligned}
 l(T(\gamma)) &= \int_a^b \frac{\sqrt{\left(\frac{du}{dt}\right)^2 + \left(\frac{dv}{dt}\right)^2}}{v(x(t), y(t))} dt \\
 &= \int_a^b \frac{|\gamma'(t)|}{y(t)} dt \\
 &= \int_a^b \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y(t)} dt \\
 &= l(\gamma)
 \end{aligned}$$

$T$  preserves the length of  $\gamma$   
 $\Rightarrow T$  is an isometry of  $\mathbb{H}^2$   
 $\phi: \text{PSL}(2, \mathbb{R}) \rightarrow \text{Homeo}(\mathbb{H}^2)$   
 $\phi(\text{PSL}(2, \mathbb{R})) \subseteq \text{Isom}(\mathbb{H}^2)$

From Equations (1), (2), and (3), what we ultimately derive is the following result: the length of the transformed path  $T(\gamma)$  can be expressed as an integral. Let me rewrite this integral clearly:

$$\int_a^b \frac{\sqrt{\left(\frac{du}{dt}\right)^2 + \left(\frac{dv}{dt}\right)^2}}{v(x(t), y(t))} dt$$

Now, analyzing the numerator, we observe that it is simply  $|\gamma'(t)|$  divided by  $y(t)$ . This conclusion comes directly from the interplay of the three equations we have discussed earlier.

Thus, this integral reduces back to:

$$\int_a^b \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y(t)} dt$$

which is precisely the expression for the length of the original path  $\gamma$ . This shows that the length of the path is preserved under the transformation  $T$ .

Therefore, we have rigorously proven that  $T$  preserves the length of the curve  $\gamma$ , and hence,  $T$  is an isometry on the upper half-plane.

Consequently, we can conclude that the group  $\text{PSL}(2, R)$  acts on the upper half-plane as isometries. In fact, we have also shown that  $\text{SL}(2, R)$ , the double cover of  $\text{PSL}(2, R)$ , also acts on the upper half-plane by isometries.

Thus, the map  $\phi$ , which sends elements of  $\text{PSL}(2, R)$  to homeomorphisms of the upper half-plane, indeed has its image contained within the isometry group of the upper half-plane.

Furthermore, an important point to note here is that the map  $\phi$  is injective. This injectivity can be demonstrated in a straightforward manner, ensuring that different elements of  $\text{PSL}(2, R)$  correspond to distinct isometries.

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$$\begin{aligned} \psi : \text{SL}(2, R) &\longrightarrow \text{Isom}(\mathbb{H}^2) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto \left( z \mapsto \frac{az+b}{cz+d} \right) \\ \text{Ker } \psi &= ? \\ \psi \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \text{Id} \quad , \quad \text{Id}(z) = z \\ \frac{az+b}{cz+d} = z &\Rightarrow az+b = cz^2+dz \\ \text{i.e. } cz^2+(d-a)z-b &= 0 \quad \forall z \in \mathbb{H}^2 \\ \Rightarrow c=0, d-a=0, b=0 & \\ \Rightarrow a=\lambda, c=b=0 & \\ ad-bc=1 & \\ \Rightarrow a^2=1 & \\ \Rightarrow a=\pm 1 \Rightarrow d=\pm 1 & \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ \text{Ker } \psi &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \end{aligned}$$

Now, let us consider the group  $\text{SL}(2, R)$ , and define a map from  $\text{SL}(2, R)$  to the isometry group of the upper half-plane. What is this map? The map takes a matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and sends it to the Möbius transformation defined as  $z \mapsto \frac{az+b}{cz+d}$ . This transformation is a group homomorphism, and let's denote this map as  $\psi$ .

Now, we ask: what is the kernel of this map  $\psi$ ? To find it, let us consider what happens if  $\psi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)$  is the identity map. For the identity map, we have  $z = z$ , which means the transformation  $\frac{az+b}{cz+d} = z$  holds for all  $z$  in the upper half-plane.

Expanding this equation, we get:

$$az + b = cz^2 + dz$$

Rearranging terms, we find:

$$cz^2 + (d - a)z - b = 0$$

This equation must hold for all  $z$  in the upper half-plane, which implies that the coefficients of each power of  $z$  must vanish. Thus, we obtain the following conditions:

- $c = 0$
- $d - a = 0$  (i.e.,  $d = a$ )
- $b = 0$

Therefore, the matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  simplifies to  $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ . Since  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  belongs to  $SL(2, R)$ , we also have the condition  $ad - bc = 1$ . Substituting  $d = a$  and  $b = c = 0$ , we get  $a^2 = 1$ , which implies that  $a = \pm 1$ .

Thus, the matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  can either be the identity matrix  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  or -1 times the identity matrix  $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ .

Hence, we have proven that the kernel of  $\psi$  is the subgroup consisting of these two matrices, the identity matrix and -1 times the identity matrix.

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$SL(2, \mathbb{R}) \xrightarrow{\psi} \text{Isom}(\mathbb{H}^2)$   
 $\downarrow \quad \searrow \phi$   
 $PSL(2, \mathbb{R})$

$\phi \circ \eta = \psi$   
 $\phi$  is a monomorphism (i.e.  $\phi$  is injective homomorphism)  
 $\Rightarrow PSL(2, \mathbb{R})$  is isomorphic to a subgroup of  $\text{Isom}(\mathbb{H}^2)$ .

Thus, we arrive at an important conclusion regarding  $SL(2, R)$ . We define the quotient map from  $PSL(2, R)$ , which we denote as  $\psi$ , alongside another map we refer to as  $\varphi$ . Notably, these two maps commute, meaning that when we compose them, we find that  $\varphi$  equals  $\psi$ .

From our discussion, we can assert that  $\varphi$  is, in fact, a monomorphism, which means that  $\varphi$  is an injective homomorphism. This injectivity implies that  $PSL(2, R)$  is isomorphic to a subgroup of the isometry group of the upper half-plane. So, I will stop here. So, in the next lecture we will learn about the geodesics of this upper half plane.