An Introduction to Hyperbolic Geometry

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Lecture – 28

Hyperbolic Metrics on Punctured Tori and Their Fuchsian Fundamental Groups

Hello! In our previous discussions, we established that a closed orientable surface with genus greater than or equal to 2 admits a hyperbolic metric. We also noted that the fundamental group of such a surface is classified as a Fuchsian group. In today's lecture, we will explore another fascinating case: the punctured torus. We will demonstrate that the punctured torus also admits a hyperbolic metric. Furthermore, its fundamental group, which is a free group of rank 2, is likewise a Fuchsian group. So, let us begin.

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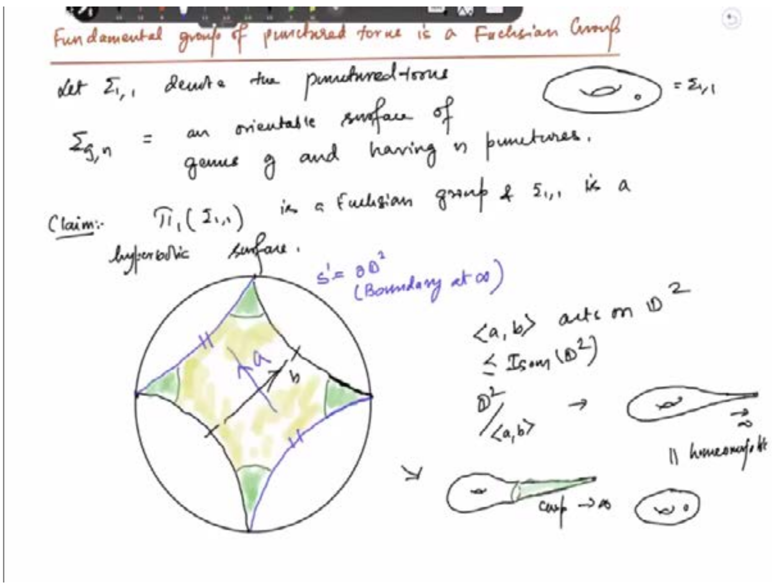

In today's session, we will demonstrate that the fundamental group of the punctured torus acts on the disc model of the hyperbolic plane. This action is characterized as a covering action, which is free and properly discontinuous. Consequently, we can conclude that the fundamental group of the punctured torus is indeed a Fuchsian group.

Let us denote the punctured torus as $\sum_{1,1}$. To visualize this, imagine a torus from which a single point has been removed, this is our punctured torus. The notation $\sum_{g,n}$ refers to an orientable surface of genus g with n punctures. Therefore, $\sum_{1,1}$ indicates a surface of genus 1 with one puncture.

Now, let’s explore how to derive a punctured torus from the unit disc model of the hyperbolic plane. We begin with the unit disc and identify four equally spaced points on its boundary. We then connect these four points with infinite geodesics, forming an ideal square whose vertices lie on the boundary, or in other words, at infinity.

Our goal here is to identify isometries that will map these blue geodesics to each other. Specifically, we seek an isometry A that maps one blue geodesic to another, and another isometry B that will take a black geodesic to a corresponding black geodesic. We will then consider the group generated by these isometries A and B.

This group acts on the disc and is a subgroup of the isometries of the unit disc equipped with a hyperbolic metric. The resulting quotient space will yield a surface that is homeomorphic to the punctured torus. To visualize this, consider that the surface will exhibit a genus of 1, with the puncture extending toward infinity.

Now, let’s explore how we arrive at this surface. If we take the ideal square and truncate it in a specific manner, essentially removing the green portion, we end up with a truncated ideal square. This truncation implies that we have deleted the green region. Next, if we focus on the yellow part of the truncated square and identify its opposite sides, we will find that one side corresponds with the other, creating a surface with one boundary component and a genus of 1.

Now, if we reintroduce the green portion back into the truncated square, we effectively restore the ideal square. In the resulting quotient space, this green section transforms into what we refer to as a cusp. This cusp represents the boundary condition of our surface.

Furthermore, it is essential to note that these two portions are asymptotic; in fact, the black geodesic and the blue geodesic are asymptotic to each other. They converge at infinity, reinforcing the structure of our quotient space.

With this visualization in mind, our next task is to identify the isometries A and B. Specifically, we will find these isometries within the context of the upper half-plane model.

We know that the upper half-plane, equipped with its Riemannian metric, is isometric to the unit disc with a similar metric. Thus, we will work within this upper half-plane framework, and let’s explore what kind of geometric configuration we will observe there.

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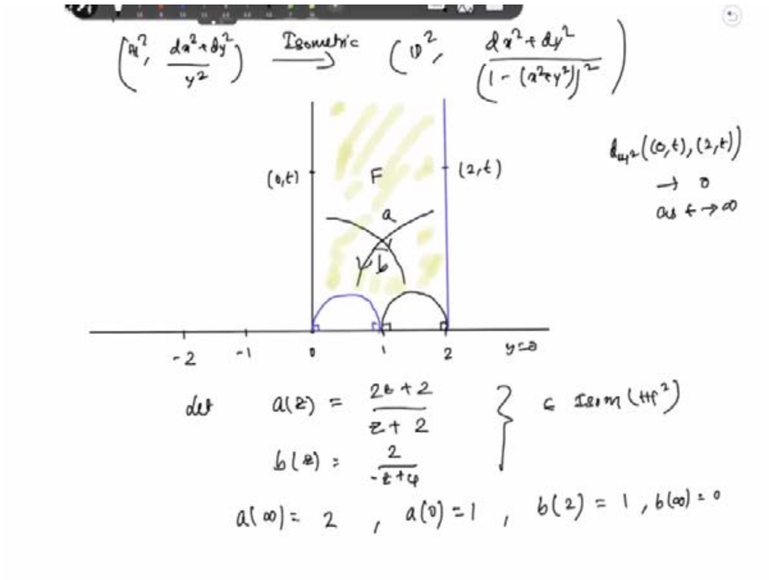

Consider the real axis, where y = 0. Here, we can visualize vertical geodesics and mark the integer points: 1, 2, -1, -2. These points correspond to the ideal square within the unit disc, and we can replicate similar structures in the upper half-plane.

For the black geodesic, we can represent it with one vertical geodesic. Additionally, we can introduce another infinite geodesic corresponding to it. Now, for the blue geodesic, we can depict another semicircle that is orthogonal to the real axis, and similarly, we can represent another blue geodesic as a vertical line.

It's crucial to note that if we select a point here, say (0, t), and another point, (2, t), the hyperbolic distance between these two points approaches zero as t tends to infinity. This means that these two vertical geodesics are asymptotic to each other.

Now, our objective is to identify the isometries A and B. The isometry A will map the vertical black geodesic to the black semicircle, while B will take the blue vertical geodesic and map it to the blue semicircle.

Here, the computation is quite straightforward. We define the transformations $a(z) = \frac{2z+2}{z+2}$ and $b(z) = \frac{2}{-z+4}$. Both a and b are isometries of the upper half-plane. In fact, if we consider the matrix corresponding to the transformation a, we will find that the determinant of that matrix equals 4. However, we can scale this matrix to obtain a representation in $\mathrm{SL}(2, R)$. Thus, both a and b are indeed isometries of the upper half-plane.

It's important to note that a(∞) = 2 and a(0) = 1. Similarly, for the transformation b, we have b(2) = 1 and b(∞) = 0. With these isometries a and b established, our next task is to verify that the group generated by a and b acts as a covering space action on the upper half-plane.

To illustrate this, we can consider the region bounded by the vertical geodesics and the two semicircles. This region will serve as our fundamental domain for the action of the group generated by a and b. Let’s denote this region as F.

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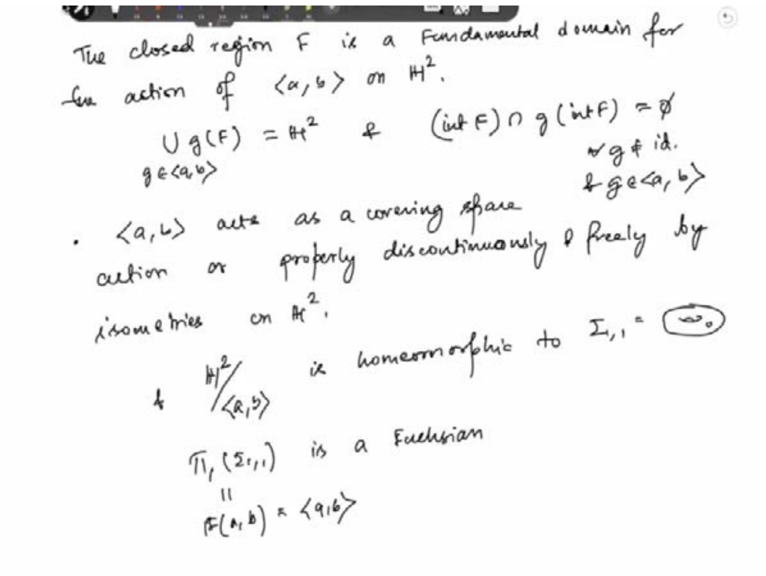

The closed region F serves as a fundamental domain for the action of the group generated by a and b on the upper half-plane. It is important to note that, because it is a fundamental region, the union of g(F) for all g belonging to the group generated by a and b will cover the entire upper half-plane. Furthermore, if we take the interior of F and translate it by any element g from the group, the intersection will be empty.

Thus, we have demonstrated that the group generated by a and b acts as a covering space action, properly discontinuously and freely, by isometries on the upper half-plane. As a result, the quotient space is homeomorphic to the punctured torus. Now, let's consider the fundamental group of this punctured torus, which is indeed a Fuchsian group. Specifically, the fundamental group of the punctured torus is the free group generated by a and b.

In summary, we have established that this free group generated by a and b is a Fuchsian group. Moreover, the fundamental group $\pi_1\left(\sum_{1,1}\right)$ is isomorphic to the group generated by a and b.

Consequently, this free group generated by a and b can be identified as a free group of rank 2, generated by a and b, which do not have any relations among themselves within this group.

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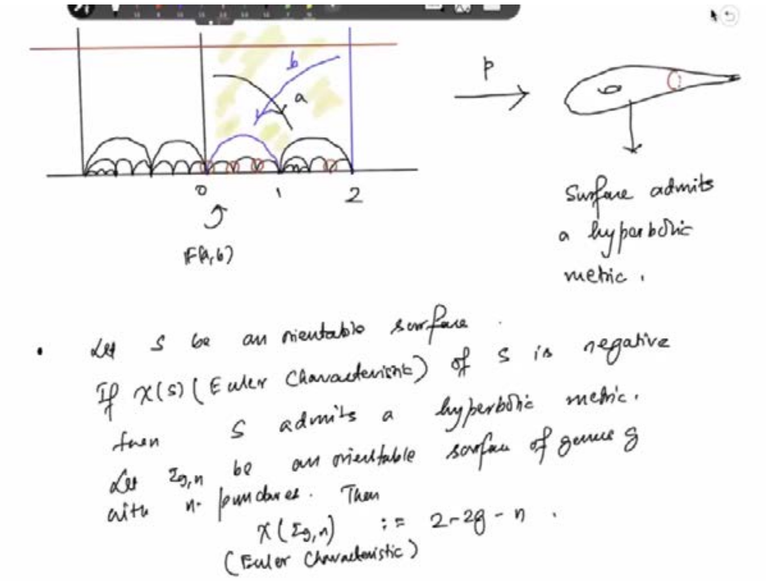

As I mentioned earlier, let's consider the upper half-plane once more. In this representation, we can label the point at the origin as 0, the next point as 1, and the point after that as 2. Now, if we introduce a vertical geodesic and another vertical geodesic, we can establish isometries a and b in this setup. This region can actually be identified as an ideal square within the upper half-plane.

Since this is a fundamental region, translating this fundamental region will yield another region, resulting in a tiling of the upper half-plane. Let me illustrate that tiling. As we apply the isometry b once more, we will obtain another ideal square, and this process can be continued indefinitely. In this manner, the free group acts on the upper half-plane; its action is free, properly discontinuous, and conducted via isometries. Consequently, the resulting quotient space will represent a surface of genus one, featuring one cusp.

Now, let’s consider a simple closed curve, which encircles this puncture, illustrated in the picture as being situated within the cusp. When we take a lift of this closed curve, we find that it becomes homotopic to a horocycle. So, if we lift that simple closed curve, denoted by the red line, it will trace a path akin to a collection of horocycles.

Therefore, we can indeed prove that this surface admits a hyperbolic metric. As established in the previous lecture, we can take the differential of the map p that describes the hyperbolic metric on the upper half-plane. Using this differential map, we can then push that hyperbolic metric onto the surface of the punctured torus.

As a generalization, let me present a key statement, though I won't provide a proof for it. Consider S to be an orientable surface. If you are familiar with the definition of the Euler characteristic, this will become clear to you. Specifically, if the Euler characteristic of S is negative, then S admits a hyperbolic metric.

Now, let me express the Euler characteristic of a surface characterized by genus g with n punctures. If you're unfamiliar with the concept of the Euler characteristic, that's perfectly fine. We can denote an orientable surface of genus g with n punctures as $\sum_{g,n}$. The Euler characteristic of this surface can be calculated using the formula:

$$\chi(S) = 2 - 2g - n$$

For example, if we set g = 1 and n = 0 (which indicates no punctures), we are simply looking at a torus. In this case, the calculation gives us:

$$\chi(\text{torus}) = 2 - 2(1) = 0$$

Thus, the Euler characteristic of a torus is 0.

Now, if we consider a surface of genus 2 with no punctures, we find:

$$\chi(S) = 2 - 2(2) = -2$$

On the other hand, if we take a genus 1 surface with n = 1 (this corresponds to a punctured torus), we compute:

$$\chi(\text{punctured torus}) = 2 - 2(1) - 1 = -1$$

This means that the Euler characteristic is negative for the punctured torus.

In summary, for any surface $\sum_{g,n}$, if 2 - 2g - n < 0, then that surface admits a hyperbolic metric. I will conclude my discussion here.