

An Introduction to Hyperbolic Geometry

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Lecture – 27

Hyperbolic Structures on Higher Genus Surfaces: Riemannian Metrics and Fuchsian Fundamental Groups

Hello! In our last lecture, we explored how the fundamental group of a closed orientable surface with genus equal to 2 is a subgroup of $PSU(1,1)$, which itself is a subgroup of the isometry group of the unit disc model of the hyperbolic plane. In today's lecture, we will build upon that discussion and prove that a closed orientable surface of genus 2 indeed admits a hyperbolic metric. So, let us begin.

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Hyperbolic Surface

$\pi_1(\Sigma_2)$ is generated by A_1, A_2, B_1, B_2 .

$\pi_1(\Sigma_2)$ is a Fuchsian group:

Let $F =$ regular octagon (closed region)

Claim:- F is a fundamental domain.

$A_1(F), A_2(F), \dots$

$g \in \pi_1(\Sigma_2), g \neq e$

$\cup_{g \in \pi_1(\Sigma_2)} g(F) = \mathbb{D}^2$ (8,8) tiling of \mathbb{D}^2

8 - Octagon

8 - (eight geodesics emerging from a vertex of the octagon)

Let us take a moment to recall what we discussed in the last class. We examined a geodesic regular octagon within the unit disc model of the hyperbolic plane. This is our unit disc, and we specifically focused on a geodesic octagon where each internal angle measures $\frac{\pi}{4}$. Thus, all these angles are $\frac{\pi}{4}$.

From this setup, we derived isometry A_1 , which maps the red geodesic to itself. Then we introduced isometry A_2 , which serves as a hyperbolic isometry taking the black geodesic to the

black geodesic. Additionally, isometry B_1 maps the blue geodesic to another blue geodesic, and finally, we have another hyperbolic isometry that takes the green geodesic to a different green geodesic.

Now, these isometries facilitate side pairings, leading to the conclusion that the fundamental group of this genus 2 surface is generated by A_1 , A_2 , B_1 , and B_2 . Consequently, this forms a finitely generated group, and in fact, it is finitely presented as well. Notably, it is a one-relator group. By identifying points using these isometries, we effectively construct the quotient that represents our genus 2 surface.

Now, let's address why this fundamental group is classified as a Fuchsian group. To explore this, let F denote our regular octagon, which is a closed region. So what exactly is F ? As mentioned, it is the octagon, and we also consider its interior. The claim here is that F serves as a fundamental domain.

First and foremost, observe that when we apply A_1 to F , it transforms into another octagon, appearing as illustrated here. Similarly, if we apply A_2 to F , it will also transition into yet another octagon.

Let's continue by examining what happens when we take any element g from the fundamental group $\pi_1(\Sigma_2)$. For our discussion, let's assume g is a non-identity element, and we apply g to the fundamental domain F . It's essential to note that the union of all the images gF , as g varies within this fundamental domain, encompasses the entirety of the unit disc. This results in what we call an 8-8 tiling of the disc.

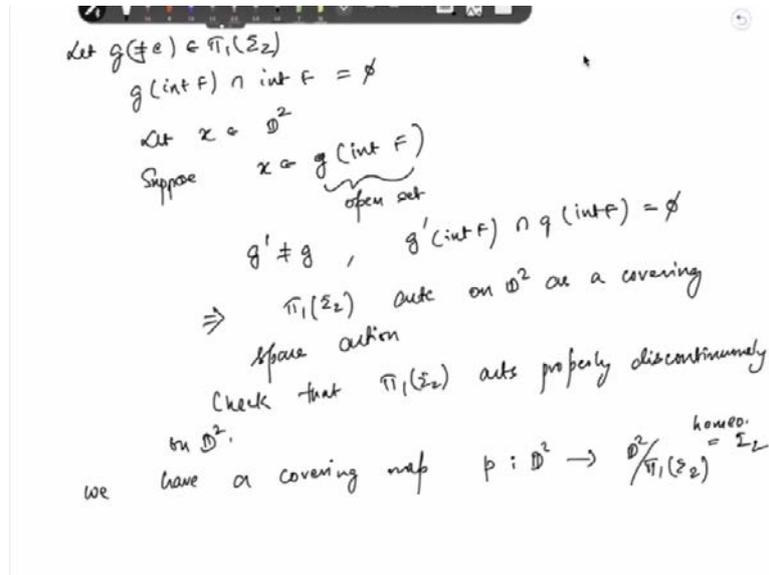
Now, what do we mean by "8-8 tiling"? I've used this term before, and it refers to a specific structure. The first "8" denotes that we are working with an octagon, specifically this regular octagon we've been discussing. The second "8" indicates that for each vertex of this octagon, eight geodesics are emanating from that point. Indeed, around any vertex, we observe eight geodesics branching out.

If we consider one of the octagons, we can clearly identify that there are eight surrounding octagons: one, two, three, four, five, six, seven, and eight, each contributing to this beautiful structure. Therefore, around each vertex, there are indeed eight octagons.

This provides us with a remarkable tiling of the unit disc. Additionally, it is crucial to

understand why F qualifies as a fundamental domain. Firstly, the union of all the gF indeed equals the unit disc. Now, let's take an arbitrary point within this disc. For instance, let's designate a point as x . If we apply some isometry, whether it be A_1, A_2, B_1, B_2 , or a product of these isometries, it will map x to another location, landing it within a different octagon. We can refer to this new point as $g(x)$. So, let me note that down.

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Let's consider any element g in the fundamental group of a genus 2 surface, specifically focusing on the case where g is not equal to the identity. In this scenario, it can be established that the intersection of the interior of F with g applied to the interior of F is empty. This statement holds true for all non-identity elements. Consequently, we can conclude that F serves as a fundamental domain.

Now, let's delve into the specifics of the interior of F . This interior represents an open set that contains the point x . To clarify, let's express this more formally: suppose x belongs to the unit disc, and let's assume that x is contained within some translate of the interior of F . Since this is an open set, it includes x as a member.

Next, consider any other element g' that is not equal to g . When we examine the intersection of g' applied to the interior of F with g applied to the interior of F , we find that this intersection is indeed empty. This leads us to an important conclusion: the fundamental group $\pi_1(\Sigma_2)$ acts on the unit disc D^2 in a manner akin to a covering space action.

Moreover, one can further establish that $\pi_1(\Sigma_2)$ acts properly discontinuously on this unit disc. From our illustrations, it's evident that $\pi_1(\Sigma_2)$ is actively acting on the space. The resulting quotient space will be homeomorphic to the genus 2 surface Σ_2 . Additionally, we have a covering map p that maps from this unit disc to our genus 2 surface. This covering map effectively connects the unit disc model to the quotient space, which, importantly, is also homeomorphic to Σ_2 .

In fact, this fundamental concept can be generalized to apply to any closed orientable surface of genus g , where g is greater than or equal to 2.

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Theorem (Poincaré) Let $g \geq 2$ and Σ_g be a closed orientable surface of genus g . Then $\pi_1(\Sigma_g)$ acts on \mathbb{D}^2 (with hyperbolic metric) freely and properly discontinuously. Hence, $\pi_1(\Sigma_g)$ is a Fuchsian Group.
 (Reference: S. Katok (Fuchsian Groups))

$\Sigma_g \cong \mathbb{D}^2 / \pi_1(\Sigma_g)$

$p: \mathbb{D}^2 \rightarrow \Sigma_g$ covering map

p is a differentiable map & p is also a local diffeomorphism

Let $x \in \mathbb{D}^2$

$d_x p: T_x(\mathbb{D}^2) \rightarrow T_{p(x)}(\Sigma_g)$, $\Sigma_g = \mathbb{D}^2 / \pi_1(\Sigma_g)$

is the differential of p at x .

$T_x(\mathbb{D}^2) = \mathbb{R}^2$, $T_{p(x)}(\Sigma_g) = \mathbb{R}^2$.

Let's formalize this concept as a theorem, primarily attributed to Poincaré. Let g be greater than or equal to 2, and let Σ_g represent a closed orientable surface of genus g . The fundamental group $\pi_1(\Sigma_g)$ acts on the unit disc equipped with a hyperbolic metric in a manner that is both free and properly discontinuous. Here, "freely" indicates that there are no fixed points in this action. Once we establish that this action corresponds to a covering space action, it follows that the action is also Fuchsian. Therefore, we conclude that $\pi_1(\Sigma_g)$ is indeed a Fuchsian group. For further reading, you may refer to the book by S. Katok titled Fuchsian Groups.

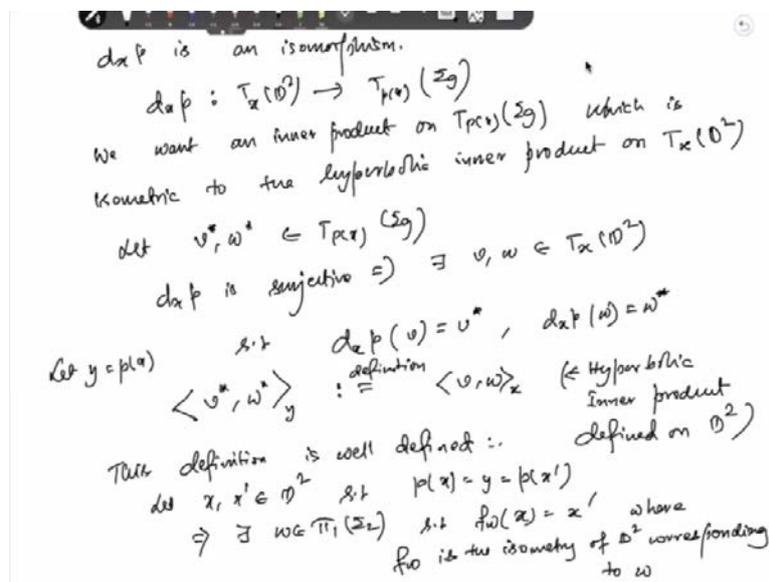
Now, let's explore how to obtain a hyperbolic metric on the genus g surface, where g is greater than or equal to 2. Again, we state that Σ_g is homeomorphic to this quotient space, and we have a covering map denoted as p that maps from the unit disc to this quotient space, which is

homeomorphic to Σ_g .

Moreover, since $\pi_1(\Sigma_g)$ acts by isometries on the unit disc, these isometries are differentiable maps. Consequently, Σ_g will form a differentiable manifold. The covering map p is a differentiable map as well, and because p is a covering map, it serves as a local homeomorphism. In fact, p is also a local diffeomorphism. This implies that the restriction of p to a suitable open set is a diffeomorphism onto its image.

Now, since p is a differentiable map, let us consider some point within this unit disc. The differential of this map, denoted as dp , maps from the tangent space of D^2 to the tangent space of the surface at the point $p(x)$. It is important to note that this surface is essentially $D^2/\pi_1(\Sigma_g)$. Hence, the differential dp at x provides a mapping from the tangent space of the disc, which is \mathbb{R}^2 , to the tangent space of the surface Σ_g , which is also \mathbb{R}^2 .

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In fact, since p is a local diffeomorphism, this implies that the differential dp is an isomorphism. Specifically, this differential serves as a map from the tangent space of the unit disc to the tangent space of the surface. Now, what we seek is an inner product defined on the tangent space of the surface that is isometric to the hyperbolic inner product on the tangent space of the disc, which is equipped with the hyperbolic metric.

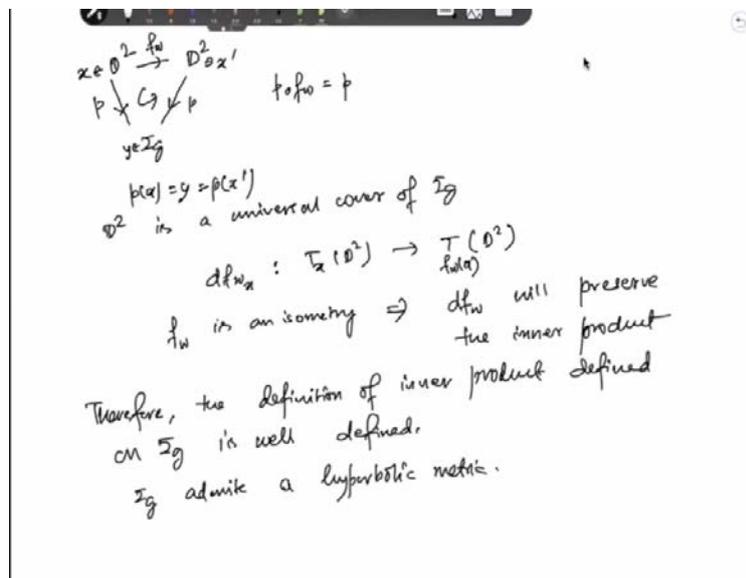
To accomplish this, let us consider two elements, v^* and w^* , from the tangent space of Σ_g . Because dp is surjective, there exist v and w in the tangent space of the unit disc at the point x

such that the differential takes v to v^* and w to w^* . Now, let's define the inner product: we will take y to be equal to $p(x)$.

For any y in Σ_g , since p is a surjective map, we can find some x such that $p(x) = y$. Thus, the inner product we define is the inner product of v and w at the point x . This inner product is specifically the hyperbolic inner product defined on the unit disc.

Now, why is this definition well-defined? By this, I mean that it is possible to find two points x and x' such that $p(x) = y$ and $p(x') = y$. Suppose x and x' are points in the unit disc satisfying $p(x) = y = p(x')$. This implies that there exists some w belonging to $\pi_1(\Sigma_g)$ such that the isometry f_w maps x to x' . Here, w corresponds to an isometry that I have referred to as f_w , which represents the isometry of the unit disc with the hyperbolic metric associated with w . This correspondence arises from the text transformation we discussed earlier.

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In this diagram, we have points x and x' , and both maps are denoted by p . Importantly, we observe that $p(x) = p(x')$, or equivalently, $p(x) = y = p(x')$. Here, the unit disc acts as a universal cover of Σ_g . Therefore, we can establish a disc transformation that takes x to x' , and this transformation allows the diagram to commute.

This transformation will indeed be a hyperbolic isometry, specifically the isometry of the unit disc that maps x to x' . As a result, the diagram commutes beautifully. Now, if we take the differential of f_w at the point x , this differential will map from the tangent space of the unit disc

at x to the tangent space of the unit disc at the point $f_w(x)$. Since f_w is an isometry, we can conclude that the differential df_w preserves the inner product. Thus, this reinforces the well-defined nature of our definition.

Consequently, the definition of the inner product on Σ_g is indeed well-defined. Furthermore, it is evident that this inner product defined on the tangent space of Σ_g is isometric to the inner product defined on the unit disc.

In fact, we can confirm that the map p is a local isometry. Therefore, we have shown that Σ_g admits a hyperbolic metric. If you are familiar with the computation of curvature, you will note that given this hyperbolic metric on the unit disc, the curvature at each point is -1 . Consequently, since p is a covering map from the unit disc to the genus g surface, it is a local isometry. Hence, the curvature at each point of Σ_g with respect to the hyperbolic metric will also be -1 .

Thus, for any closed orientable surface of genus greater than or equal to 2, the curvature at each point is indeed -1 . I will stop here.