An Introduction to Hyperbolic Geometry

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Lecture – 26

From Flat Tori to Hyperbolic Surfaces: Riemannian Metrics and Fuchsian Fundamental Groups

Hello! In the last lecture, we discussed that a torus admits a Riemannian metric, which is isometric to the Euclidean metric. This means that we can equip the torus with a Euclidean metric, and the curvature at each point with respect to this metric is zero. Now, using a similar concept, we will demonstrate that any closed orientable surface of genus $g \geq 2$ admits a hyperbolic metric.

What this means is that we can place a Riemannian metric on any closed surface of genus $g \geq 2$ such that the Riemannian metric is isometric to a hyperbolic metric. Additionally, we will show that the fundamental group of any closed orientable surface of genus $g \geq 2$ is a Fuchsian group. Specifically, we will prove that if you take a genus 2 surface, which is both closed and orientable, it will admit a hyperbolic metric, and its fundamental group will be a Fuchsian group.

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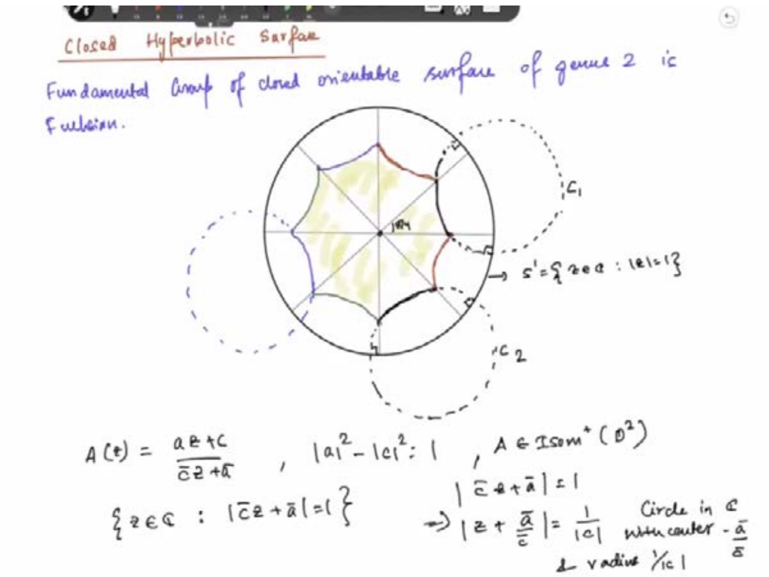

The same reasoning applies to surfaces of higher genus, meaning the result holds for any surface with $g \geq 2$. This result is known as Poincaré's theorem. However, I won't be proving the entire theorem in full generality; instead, I will focus on proving it for the case of genus g = 2.

First, we will prove that the fundamental group of any closed orientable surface with genus g ≥ 2 is indeed a subgroup of $\mathrm{PSL}(2, R)$ or $\mathrm{PSU}(1,1)$. Subsequently, we will demonstrate that this fundamental group is a Fuchsian group. This means that the fundamental group of the surface will act on the unit disk model of the hyperbolic plane in a properly discontinuous manner.

Establishing this will confirm that the fundamental group of the surface is a Fuchsian group. Utilizing the same principle, if you take any closed orientable surface with genus g ≥ 2, then its fundamental group will also be Fuchsian. This is because the fundamental groups act properly discontinuously on the unit disk model of the hyperbolic plane, and their actions are also free. Consequently, the quotient space will yield a surface with genus g ≥ 2, which will admit a hyperbolic metric. This metric will originate from the hyperbolic plane through the covering map.

Let us now focus on the case of a genus 2 surface. As we have already established, there exists a regular octagon in the unit disk whose sides are geodesics. This is our regular octagon. The angles at each vertex of this octagon measure $\frac{\pi}{4}$, a detail we have previously discussed.

Now, what we will do is suitably identify the sides of this octagon. The resulting quotient space from these identifications will yield a genus 2 surface. For instance, the red geodesic will be identified with another red geodesic, and this identification will be achieved through a hyperbolic isometry. Similarly, the black geodesic will be identified with another black geodesic.

The blue geodesics will be identified with other blue geodesics, and the green geodesics will be identified with their corresponding green geodesics. Once we make these identifications, the quotient space will indeed become a genus 2 surface. Now, our first task is to find an isometry that will map the first black geodesic, denoted as c_{12}, to the second black geodesic, denoted as c_2.

We know that this geodesic is part of a circle, which can be extended until it intersects the boundary of the unit disk orthogonally. Let us denote this circle as c_1. The intersection of c_1

with the unit circle S^1 occurs orthogonally. Similarly, we can extend this geodesic to create infinitely many geodesics that will also intersect orthogonally. Further extending this in the complex plane, we will derive c_2, which will intersect the circle S^1 orthogonally as well.

Now, our objective is to find an isometry that takes c_1 to c_2. In the unit disk model of the hyperbolic plane, any isometry can be represented in the following form:

$$\frac{az + c}{\bar{c}z + \bar{a}}$$

Here, it is crucial that the condition $|a|^2 - |c|^2 = 1$ holds, indicating that we are dealing with an orientation-preserving isometry of the unit disk.

Next, let us consider the circle defined by $|\bar{c}z + \bar{a}| = 1$. If we analyze this expression, we can rewrite it as:

$$\left|\frac{\bar{c}z+\bar{a}}{c}\right| = 1$$

This indicates that the modulus of $z + \bar{a}/\bar{c}$ is equal to 1/|c|. Consequently, we have a circle in the complex plane with its center located at $-\bar{a}/\bar{c}$ and a radius of 1/|c|.

Now, let us make another observation. Suppose z satisfies this condition; in that case, z belongs to the set of complex numbers z such that $|\bar{c}z + \bar{a}| = 1$.

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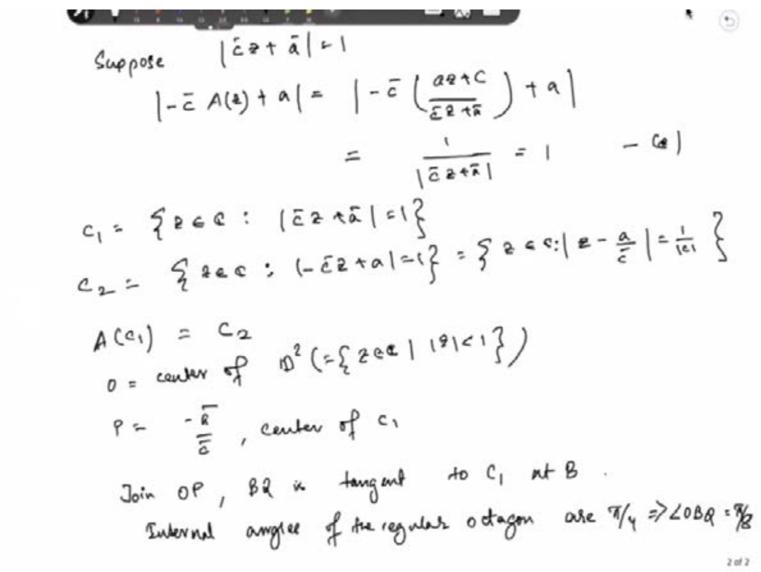

Let’s suppose that $|\bar{c}z + \bar{A}| = 1$. Now, if I compute the modulus of the expression $|-\bar{c}\bar{A}z + A|$, we can rewrite this as:

$$|-\bar{c}\bar{A}z + A| = |-\bar{c} \cdot az + c| \quad \text{"}\{(substituting^{-}\{A\}\, for\ a)\}$$

By applying the properties of modulus, this becomes:

$$= |-\bar{c}az + c| = \left|-\bar{c}\left(z - \frac{a}{\bar{c}}\right)\right|$$

This implies that the expression is equal to:

$$\frac{1}{|\bar{c}z + \bar{A}|} = 1$$

So, this observation leads us to an important conclusion. If I designate c_1 as the circle defined by z in C such that $|\bar{c}z + \bar{A}| = 1$, and c_2 as the circle defined by z in C such that $-\bar{c}z + A = 1$, we see that c_2 is also a circle, with its center at $\frac{A}{\bar{c}}$ and a radius of $\frac{1}{|c|}$.

This equation demonstrates that the image of c_1 under the mapping A corresponds to c_2. Consequently, we have established that $A(c_1)$ is a subset of c_2. Since A is a Möbius transformation, it will indeed map $A(c_1)$ to c_2.

Now, we are tasked with finding an isometry that takes c_1 to c_2. There exists a Möbius transformation that achieves this mapping, and our next step is to determine the coefficients a and c involved in this transformation.

To aid in this process, let’s introduce a notation: let O represent the center of the unit disk, which is simply the origin. Let us define point P to be $-\frac{\bar{A}}{\bar{c}}$, which serves as the center of c_1.

Next, we will connect points O and P. Suppose we have point P defined, and we now take this point to be B. By considering the line segment joining O and P, we form a triangle where the point O is the origin.

Now, we need to consider what the angle at this point will be. This angle measures $\frac{\pi}{4}$. But why is this angle $\frac{\pi}{4}$? Let's delve into the reasoning behind it.

Now, let’s proceed by drawing a tangent line at point B to the circle c_1. This tangent line forms a right angle with the radius at point B, which means this angle measures 90° or $\frac{\pi}{2}$. Additionally, it's important to note that the angle at point P, which we just discussed, is half of $\frac{\pi}{4}$; thus, it measures $\frac{\pi}{8}$.

We also established earlier that the regular octagon has internal angles of $\frac{\pi}{4}$. Therefore, considering this angle at P of $\frac{\pi}{4}$, the adjacent angle at point B is indeed $\frac{\pi}{8}$. As a result, the angle we are considering will total $\frac{3\pi}{4}$.

Next, let's denote point Q as the intersection where this tangent meets. Now, if we examine triangle PBQ, which is a right triangle at point P, we observe that one of its angles is $\frac{\pi}{4}$, and consequently, the angle QPB will also measure $\frac{\pi}{4}$.

To summarize our findings: we have connected points O and P as previously discussed. The tangent line BQ is tangent to circle c_1 at point P, and the internal angles of the regular octagon are indeed $\frac{\pi}{4}$. Thus, we can conclude that the angle OBQ measures $\frac{\pi}{8}$.

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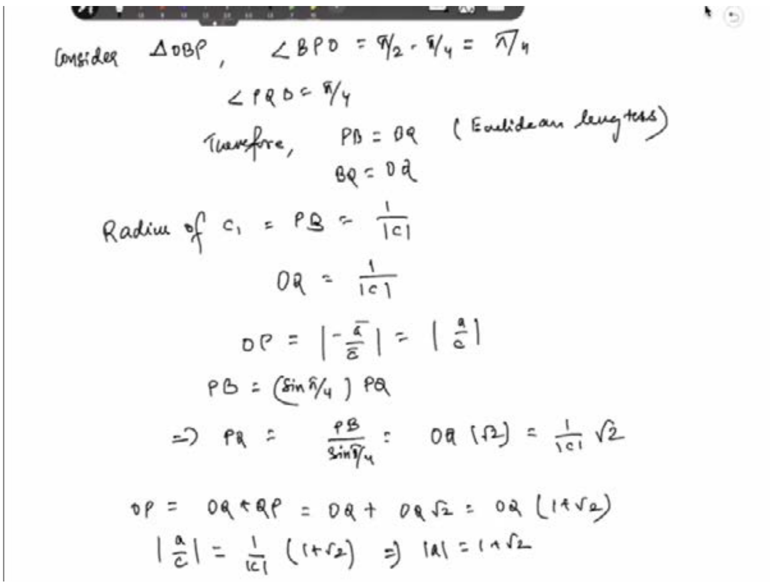

Now, let's examine triangle OBP. The angle BPO can be calculated as $\frac{\pi}{2} - \frac{\pi}{4}$, which simplifies to $\frac{\pi}{4}$. This means that the angle PQB is also $\frac{\pi}{4}$. From this configuration, since angles BQP and BPQ are equal, we can conclude that the lengths of sides PB and PQ must be the same.

In this context of Euclidean geometry, we thus have PB = BQ. These represent the lengths in Euclidean space. Furthermore, since BQ is also equal to OQ, in triangle OBQ, we have two equal angles, which implies OQ = BQ.

Now, let's discuss the radius of circle c_1. It is given by $\frac{1}{|c|}$, and this is equivalent to the length of PB. Therefore, we can affirm that the radius of c_1, denoted as PB, is indeed $\frac{1}{|c|}$. Consequently, we find that OQ is also equal to $\frac{1}{|c|}$.

Next, we know the center P can be expressed as $-\bar{a}/\bar{c}$, which translates to $\frac{a}{|c|}$. It's also worth noting that PB can be represented as $\sin\left(\frac{\pi}{4}\right) \times PQ$. This means we have:

$$PB = \sin\left(\frac{\pi}{4}\right) \times PQ$$

From this, we can derive that:

$$PQ = \frac{PB}{\sin\left(\frac{\pi}{4}\right)}$$

Substituting in our earlier value for PB, we find:

$$PQ = OQ\sqrt{2} = \frac{1}{|c|}\sqrt{2}$$

Now, let's look at the relationship between OP, OQ, and QP. We have:

$$OP = OQ + QP = OQ + OQ\sqrt{2} = OQ\left(1 + \sqrt{2}\right)$$

Thus, we can express $|a|/|c|$ as:

$$\frac{|a|}{|c|} = \frac{1}{|c|}\left(1 + \sqrt{2}\right)$$

This leads us to the conclusion that:

$$|a| = 1 + \sqrt{2}$$

Now, let's consider the centers of circles c_1 and c_2. Referring back to our diagram, the center of c_1 is point P, while we define the center of c_2 to be $\frac{a}{\bar{c}}$. This establishes a relationship between the centers of c_1 and c_2, that is crucial for our further analysis.

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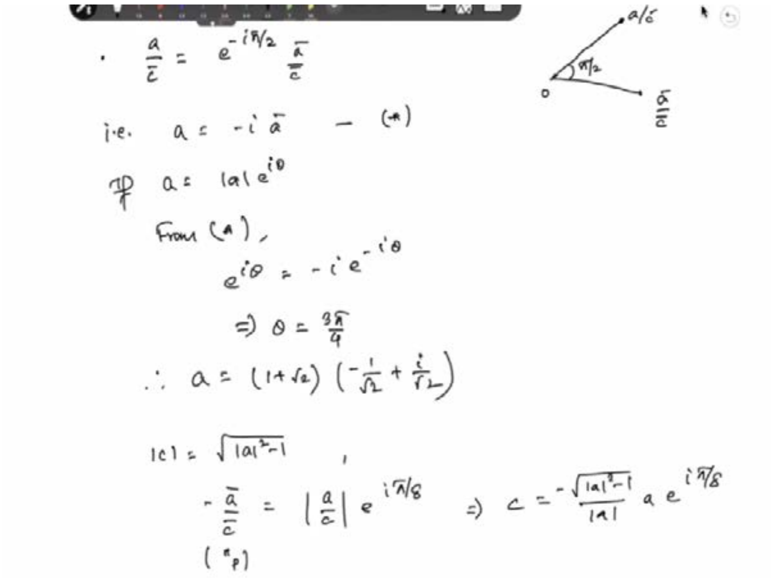

The expression $\frac{a}{\bar{c}}$ can be rewritten as $e^{-i\frac{\pi \bar{a}}{2\,\bar{c}}}$. When we visualize this, we observe that both points are equidistant from the origin, with an angle of $\frac{\pi}{2}$ between them. Hence, we conclude that $\frac{a}{\bar{c}} = e^{-i\frac{\pi \bar{a}}{2\,\bar{c}}}$, which simplifies to $a = -i\bar{a}$.

Let's label this equation as our star equation. Now, suppose we express a in polar form as $a = |a|e^{i\theta}$. From the star equation, we derive that $e^{i\theta} = -ie^{-i\theta}$. This leads us to conclude that θ must be equal to $\frac{3\pi}{4}$.

Previously, we determined that $|a| = 1 + \sqrt{2}$. Therefore, we can write a as:

$$a = (1+\sqrt{2})e^{i\frac{3\pi}{4}} = (1+\sqrt{2})\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

Next, we need to find |c|. This can be expressed as $|c| = \sqrt{|a|^2 - 1}$. Furthermore, we have established that:

$-\frac{\bar{a}}{\bar{c}} = |-\frac{\bar{a}}{\bar{c}}| = \frac{a}{c}$

This expression makes an angle of $\frac{\pi}{8}$ with the x-axis, confirming that $-\frac{\bar{a}}{\bar{c}}$ represents the point P in the complex plane, positioned at an angle of $\frac{\pi}{8}$ relative to the real axis.

From this information, we can derive c as follows:

$$c = -\frac{\sqrt{|a|^2 - 1}}{|a|} e^{i\frac{\pi}{8}}$$

Now that we know the values for a and |a|, we can compute c as well. Thus, we have successfully established the values for both a and c.

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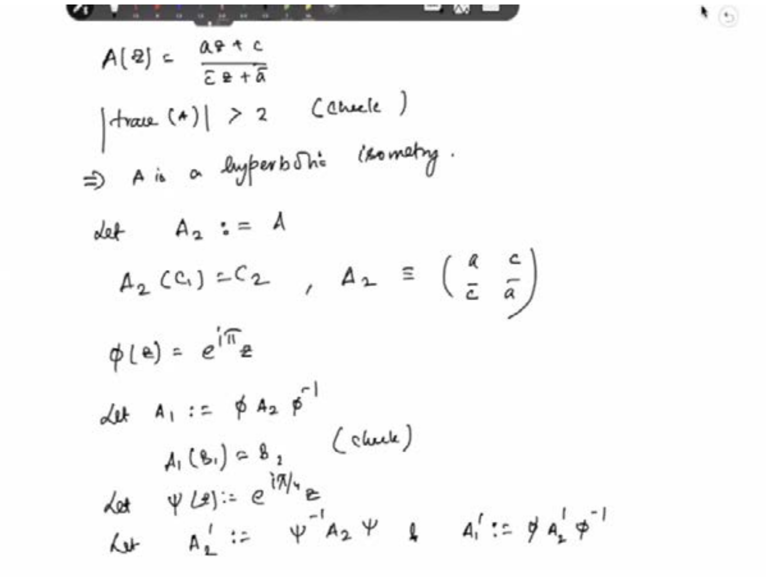

Let $A(z) = \frac{az+c}{\bar{c}z+\bar{a}}$. Upon examination, we can observe that the modulus of the trace of this matrix is greater than 2, which is a characteristic property. This indicates that A is indeed a hyperbolic isometry. Now, let's denote this isometry as A_2, which maps the circle c_1 to c_2. We have already calculated the matrix representation of A.

Next, we consider another isometry defined as $\phi(z) = e^{i\pi}z$. This function represents a rotation by an angle of π in the unit disc model of the hyperbolic plane. Importantly, this isometric transformation fixes the origin.

Now, let's define A_1 as the conjugate of A_2 by φ, expressed as $A_1 = \phi A_2 \phi^{-1}$. The action of A_1 will map the blue geodesic B_1 to the blue geodesic B_2. To illustrate this, we can refer to the corresponding circle of B_2 as c_2', drawn in correspondence with B_2. Thus, A_1 effectively transforms B_1 to B_2, and we can verify this transformation.

Now, let's introduce another rotation by an angle of $\frac{\pi}{4}$ around the origin, denoted by ψ(z). We will refer to the conjugate of A_2 by ψ as A_1', defined as $A_1' = \psi^{-1}A_2$. It is crucial to clarify that A_1' is indeed the conjugate of A_2 by ψ, not φ.

This notation is essential, and it's important to ensure we are clear about which conjugate we are referring to. We can proceed to check the properties of this transformation as needed.

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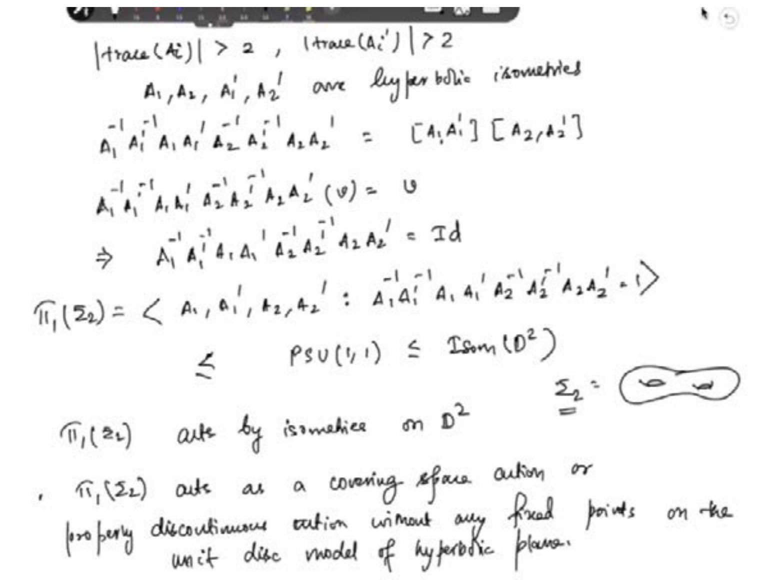

The trace of each A_i, when we take the modulus, is greater than 2. Similarly, the modulus of the trace of A'_i is also greater than 2. This confirms that all transformations A_1, A_2, A_1', and A_2' are indeed hyperbolic isometries.

Now, let's consider the product of the commutator. Specifically, we take the following sequence: first A_1^{-1}, then $A_1'^{-1}$, followed by A_1', then A_2^{-1}, and finally $A_2'^{-1}$ and A_2'. This sequence represents the product of the commutators, which can be expressed as $[A_1, A_1']$ and $[A_2, A_2']$.

Now, let's refer back to our diagram. If we denote a specific vertex as v and apply the isometry represented by the product of these commutators, we observe that it remains isometric. In fact, if we apply this transformation to vertex v, we find that it maps back to itself. This indicates that this isometry fixes an interior point of the unit disc.

Since it fixes a point in the interior, it cannot be classified as an elliptic isometry; therefore, it must be the identity element. Consequently, the group generated by A_1, A_1', A_2, and A_2' forms a subgroup of the isometries of the unit disc. This relation reinforces the structure and nature of the isometries we are working with.

What we have demonstrated is that this indeed forms a subgroup of PSU(1,1), which itself is a subgroup of the isometries of the unit disc model of the hyperbolic plane. Remarkably, this group corresponds precisely to the fundamental group of a genus 2 surface. Here is our genus 2 surface.

So far, we have identified four isometries. The first, denoted as A_1, maps the black geodesics onto themselves. The second isometry, A_2, takes B_1 to B_2. Additionally, one of the isometries, either A_1' or A_2', will map the green geodesics to green geodesics, while the other will map the red geodesics to red geodesics.

In total, we have established four geodesics responsible for performing side pairings, and we have only one relation: the product of the commutators equals 1. This leads us to conclude that we are observing the fundamental group of this genus 2 surface.

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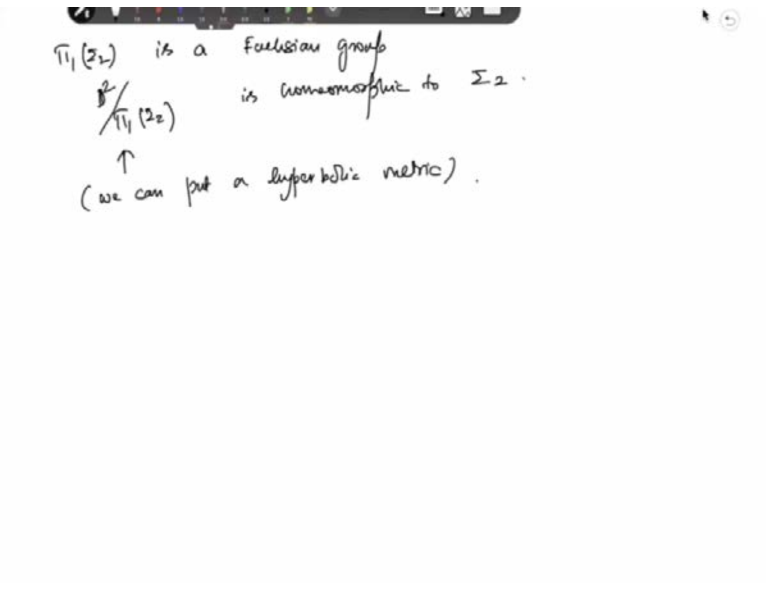

We constructed this genus 2 surface from a regular octagon by employing side pairings. Notably, this fundamental group of the genus 2 surface acts via isometries on the unit disc model of the hyperbolic plane. In fact, one can demonstrate that this action functions as a covering space action, or more precisely, as a properly discontinuous action without any fixed points on the unit disc model of the hyperbolic plane. This illustrates the elegant interplay between geometry and topology in our study.

This will establish that the fundamental group $\pi_1(\Sigma_2)$ is indeed a Fuchsian group. To summarize our findings, when we consider the unit disc equipped with a hyperbolic metric, we see that $\pi_1(\Sigma_2)$ acts properly discontinuously on this disc. Consequently, the resulting quotient space is homeomorphic to Σ_2. Furthermore, within this space, we can endow it with a hyperbolic metric. I will stop here, and in our next class, I will demonstrate how to impose this hyperbolic metric on the genus 2 surface.