An Introduction to Hyperbolic Geometry

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Module - 7

Lecture – 25

Riemannian Structures and Covering Spaces: From Flat Tori to Hyperbolic Surfaces

Hello. In this lecture, we will first demonstrate that the torus admits a Euclidean structure. Now, what does this mean? Essentially, it means that for any given torus, we can assign a Riemannian metric that is isometric to the Euclidean metric. If we manage to do this, the torus, with this Euclidean metric, will have zero curvature at every point. Even though the torus is not homeomorphic to the Euclidean plane, its curvature will still be zero at each point.

Therefore, we can classify the torus as a constant curvature surface where the curvature at every point is precisely zero. This implies that it's possible to define a metric on the torus so that its curvature remains zero everywhere.

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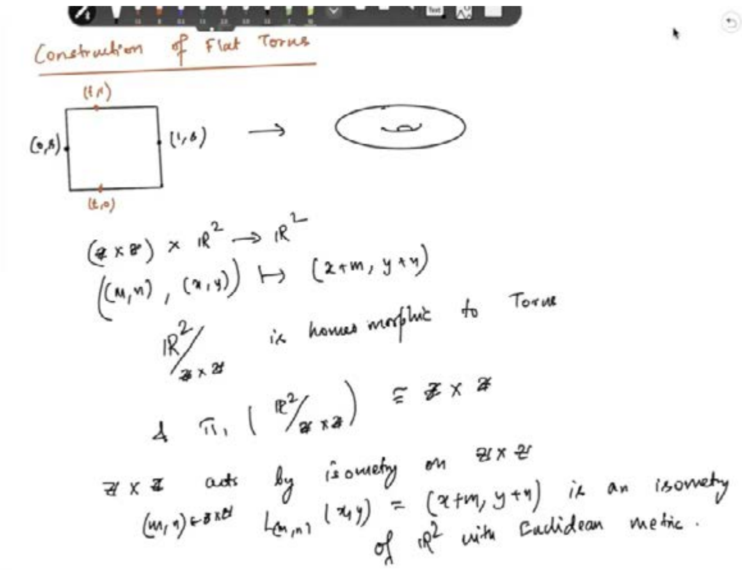

Using this same principle, we will now move towards finding a hyperbolic surface. As discussed in the last lecture, we can take a regular 4g-gon whose sides are geodesics, where $g \geq 2$. This 4g-gon can be isometrically embedded into the hyperbolic plane. By performing specific side pairings, similar to the process used for the torus, we will construct a hyperbolic surface. So, let us begin.

Let’s recall how we construct a torus from a unit square. Imagine we start with a unit square, and to form the torus, we identify the opposite sides. So, this side on the left is identified with the side on the right, meaning that a point (t, 0) on one edge is identified with the point (t, 1) on the opposite edge. Similarly, for a point at the bottom, say (0, s), this point is identified with (1, s) on the top. After this identification, what you get is the structure of a torus.

Now, looking at it from a different perspective, the torus can also be constructed through the action of the group $Z \times Z$ on R^2. Specifically, this group acts on R^2 by translations. We’ve already seen this process: the action of $Z \times Z$ is properly discontinuous, which means it’s a covering space action. The resulting quotient space is homeomorphic to the torus, and the fundamental group of this quotient space is isomorphic to $Z \times Z$.

But there's more to this. The action of $Z \times Z$ on R^2 is not just a simple group action, it acts by isometries. This means that if you take any element $(m, n) \in Z \times Z$, and consider the corresponding translation map, this map is an isometry of R^2 equipped with the Euclidean metric.

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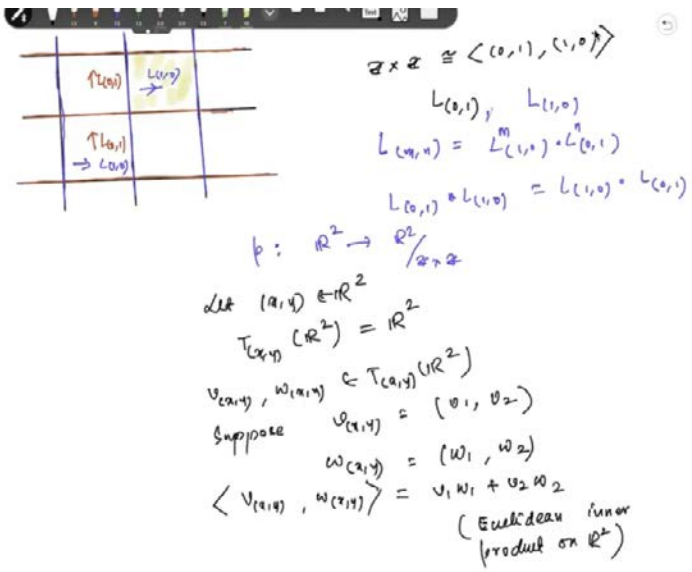

Let's revisit the situation by drawing the lattice in R^2, representing the group $Z \times Z$. This lattice is generated by the two vectors (0, 1) and (1, 0). Now, what does the transformation $L_{0,1}$ do? It takes any horizontal geodesic in the plane and translates it to the next horizontal geodesic by a unit length. Essentially, it shifts the entire set of horizontal geodesics by a distance of 1. Similarly, the isometry $L_{1,0}$ translates the vertical geodesics by a vector (1, 0), shifting them by a unit distance vertically.

Now, you can generalize this. The transformation $L_{m,n}$ can be written as a composition: $L_{m,n} = L_{1,0}^{m} \cdot L_{0,1}^{n}$. And importantly, these two isometries commute with each other, meaning the order of applying these transformations doesn't matter. All of these translations correspond to the deck transformations associated with the covering map.

So, what do we have at this point? We have a covering map from R^2 to the quotient space $R^{\mathbb{2}}/(Z \times Z)$. The identification of boundary points in this fundamental region is performed through isometries. This process allows us to transfer the Euclidean metric from R^2 to the quotient space $R^{\mathbb{2}}/(Z \times Z)$.

Now, let’s consider a point in R^2. The tangent space at this point is, naturally, still R^2. Take two vectors in this tangent space, say $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$. The Euclidean inner product between these vectors is simply $v_1 w_1 + v_2 w_2$, which defines the standard Euclidean inner product on R^2.

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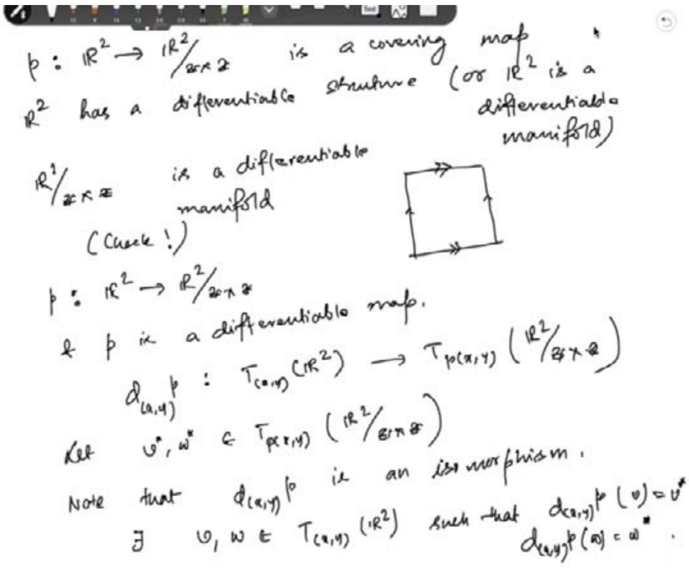

Now, let's consider the covering map p. Since R^2 has a differentiable structure, in fact, we can say that R^2 is a differentiable manifold. You can check that $R^{\mathbb{2}}/(Z \times Z)$, the quotient space, is also a differentiable manifold. Why do we say this? Essentially, you obtain the torus by identifying opposite sides of a square, and these gluing maps are isometries; more precisely, they are diffeomorphisms. So, when you construct the quotient space using these diffeomorphisms, the result is a differentiable manifold. You can verify this step for rigor.

Now, we have a map p from R^2 to $R^2/(Z \times Z)$. Both R^2 and the quotient space $R^2/(Z \times Z)$ are differentiable manifolds, and the map p itself is differentiable. If you take the differential of p at a point (x, y), this gives a map from the tangent space of R^2 at (x, y) to the tangent space of $R^2/(Z \times Z)$ at the point p(x, y).

Now, let's focus on two vectors, say v^* and w^*, in the tangent space of the torus (i.e., $R^2/(Z \times Z)$). Also, recall that the differential map is linear. Since p is a local diffeomorphism (meaning it's locally a homeomorphism), and the transition or gluing maps are simply translations, they are also differentiable. Hence, the map p is differentiable, and its differential is an isomorphism.

Given that the differential dp is an isomorphism, there exist vectors v and w in the tangent space of R^2 such that the differential $dp(v) = v^*$ and $dp(w) = w^*$. Now, to define a Riemannian metric on the torus (which gives an inner product on the tangent space of the torus), we use the vectors v^* and w^* from the tangent space of the torus. Through this process, we will derive a suitable inner product, completing the construction of the Riemannian metric on the torus.

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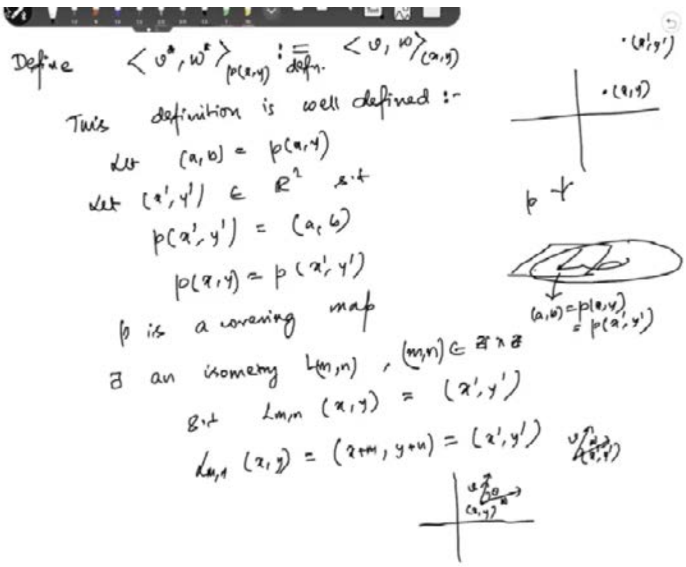

Now, let us define the inner product at the point p(x, y). This inner product is simply the

Euclidean inner product of the vectors v and w at the point (x, y). So, corresponding to the vector v^* in the tangent space of the torus, we have a vector v in R^2, and similarly, corresponding to w^*, there exists a unique vector w in R^2. You just take the inner product of these vectors in R^2.

Now, why is this definition of the inner product well-defined? By "well-defined," I mean that the result is consistent regardless of how we represent the point p(x, y). Let's explore this. Suppose we have another point (x', y') such that p(x', y') = p(x, y) = (a, b). In other words, both (x, y) and (x', y') map to the same point (a, b) on the torus. So, the map p takes both of these points in R^2 to the same point on the quotient space, the torus.

Since p is a covering map, the points (x, y) and (x', y') differ by an isometry of R^2. More specifically, there exists an isometry, denoted by $L_{m,n}$, where $m, n \in Z \times Z$, such that $L_{m,n}$(x, y) = (x', y'). In fact, $L_{m,n}$ represents a translation by the integer vector (m, n), which maps (x, y) to (x', y').

Because $L_{m,n}$ is an isometry, it preserves the inner product. Thus, the inner product of the vectors v and w at (x, y) is the same as the inner product of their corresponding vectors at (x', y'), ensuring that our definition is consistent. In other words, the translation by $L_{m,n}$ does not change the inner product.

To visualize, imagine the point (x, y) in R^2, and suppose we have two vectors v and w based at this point. If we translate to the point (x', y'), which is related by the isometry $L_{m,n}$, the vectors v and w remain the same but are now based at (x', y'). Importantly, the angle between these vectors is preserved, maintaining the inner product.

Thus, we have a well-defined inner product on the torus, consistent under the identification of points in R^2.

In other words, the differential of this translation map is a transformation from the tangent space of R^2 at the point (x, y) to the tangent space of R^2 at the point (x', y'). If we consider two vectors, v and w, which belong to the tangent space of R^2 at (x, y), then their images under the differential of the isometry $L_{m,n}$, denoted as $dL_{m,n}(v)$ and $dL_{m,n}(w)$, will be the corresponding vectors in the tangent space at (x', y'). This is essentially the same process for any pair of points (x, y) and (x', y').

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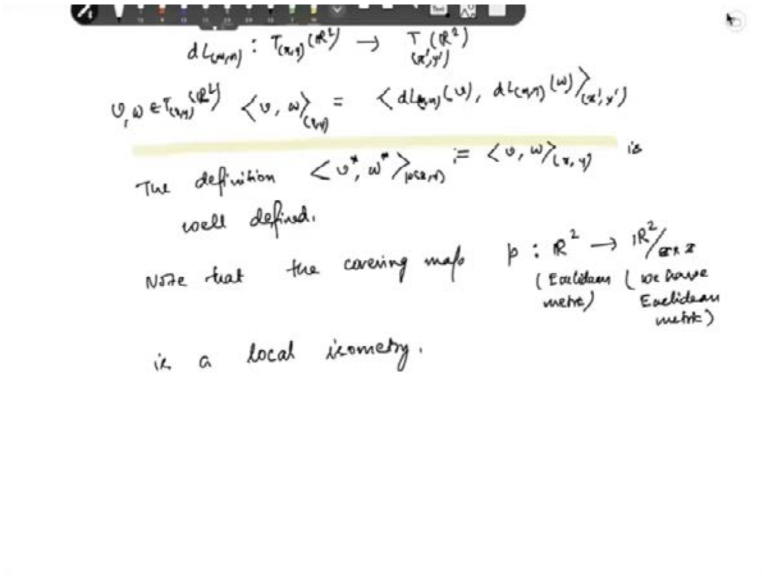

So, what are we trying to achieve? We aim to define an inner product on the torus. To do this, we begin by selecting any point on the torus, say (a, b). Now, there exists a point (x, y) in R^2 such that p(x, y) = (a, b), where p is the covering map. Next, we consider two vectors v^* and w^* in the tangent space of the torus at the point (a, b). We've already established that the inner product on the torus at this point is defined as the Euclidean inner product of the corresponding vectors v and w in R^2, where the differential dp(v) = v^* and dp(w) = w^*.

Now, suppose there is another point (x', y') such that p(x', y') = p(x, y). The inner product at (x', y') must still match that at (x, y). Hence, the inner product $\langle v *, w * \rangle$ at the point p(x, y) is well-defined, as it remains consistent regardless of the particular preimage in R^2.

Because of this, the torus admits a Riemannian metric, which in this case is isometric to the Euclidean inner product. Thus, the torus has a Euclidean metric, and the curvature at every point with respect to this metric is zero. Also, it is important to note that the covering map p from R^2 to the torus is a local isometry. Since p is a covering map, it is a local homeomorphism, and, as we have just seen, it is also a local isometry.

Using this same reasoning, we will show that the fundamental group of any closed orientable surface of genus $g \geq 2$ acts on the unit disk (which represents the hyperbolic plane) with a properly discontinuous action, also known as a covering space action. The quotient space will then be a surface of genus g. Therefore, all closed orientable surfaces of genus $g \geq 2$ will admit a hyperbolic structure since they arise from the quotient of the hyperbolic plane.

In a previous lecture, we saw that there exists a regular 4g-gon in the hyperbolic plane whose internal angles sum to 2π. Using side pairings, we can show that any surface of genus $g \geq 2$, which is also closed and orientable, can be endowed with a hyperbolic structure. I will stop here.