An Introduction to Hyperbolic Geometry

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Module - 7

Lecture – 24

Fundamental Groups of Genus g Surfaces and Their Fuchsian Representations

Hello. Our next objective is to demonstrate that the fundamental group of a closed, orientable surface of genus g, where $g \geq 2$, is a Fuchsian group. To achieve this, we need to construct a regular 4g-gon whose sides are geodesics in the hyperbolic plane. This process is quite similar to the construction of a torus from a square.

Let's recall how the torus was constructed. We took a square, which is an isometric subspace of R^2, and identified its opposite sides. This identification led us to the torus. Now, the same concept applies here, but in the context of a surface of higher genus. We will construct a regular 4g-gon in the hyperbolic plane, with all its sides as geodesics. This 4g-gon serves as an isometric subspace of the hyperbolic plane. Once we perform the appropriate side pairings, we will obtain a surface of genus g. So, the underlying philosophy remains the same: just as we identified opposite sides of a square to form a torus, we will identify sides of this regular 4g-gon to form a genus g surface. Let's begin the construction.

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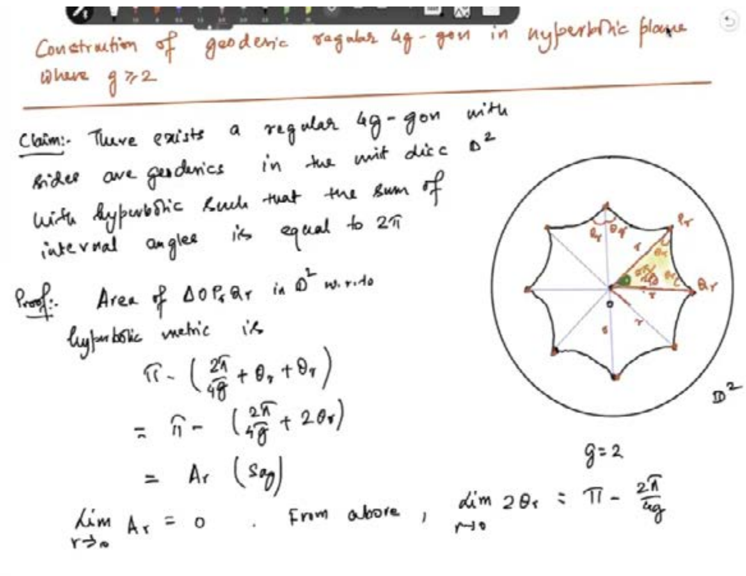

The claim is as follows: there exists a regular 4g-gon with geodesic sides in the unit disk model of the hyperbolic plane such that the sum of its internal angles equals 2π. I've sketched a diagram in the unit disk model of the hyperbolic plane, where g = 2, but this method can be applied for any $g \geq 2$.

First, let O be the origin. From here, we can draw geodesics such that the angle between two consecutive geodesics is $\frac{2\pi}{4g}$. For any $g \geq 2$, you draw two geodesics originating from the center O that form an angle of $\frac{2\pi}{4g}$. At some distance r from the origin along each geodesic, place a vertex. These vertices are equidistant in the unit disk.

Now, focus on the triangle formed by these two geodesics and a line segment connecting the two vertices. Assume that the angle at the vertex between the two geodesics is θ_r. By symmetry, the angles at both vertices of this triangle will be equal, so both will measure θ_r.

Let's label these two vertices as P_r and Q_r. What we want to show is that for some r_0, the sum of the internal angles, meaning $2\theta_r$, satisfies the relation $4g\theta_r = 2$ for that specific r_0.

Now, for the proof: the area A_r of the triangle OP_rQ_r in the unit disk, with respect to the hyperbolic metric, is given by the formula:

$$A_r = \pi - \left(\frac{2\pi}{4g} + \theta_r + \theta_r\right)$$

This simplifies to:

$$A_r = \pi - \frac{2\pi}{4g} - 2\theta_r$$

If we let $r \to 0$, the area A_r tends to 0, since the triangle collapses into a point, and the area of a point is zero. Hence, we have:

$$\lim_{r \to 0} A_r = 0$$

From the previous equation for A_r, this implies that:

$$\lim_{r \to 0} 2\,\theta_r = \pi - \frac{2\pi}{4g}$$

Note that earlier I mistakenly used $\frac{2\pi}{g}$ instead of $\frac{2\pi}{4g}$, so please adjust that accordingly in the equations. The correct relation is based on the 4g-gon, so the angle should be $\frac{2\pi}{4g}$.

Therefore, when $r \to 0$, we get:

$$2\theta_r = \pi - \frac{2\pi}{4g}$$

Next, let's introduce S_r, which allows us to proceed with further calculations.

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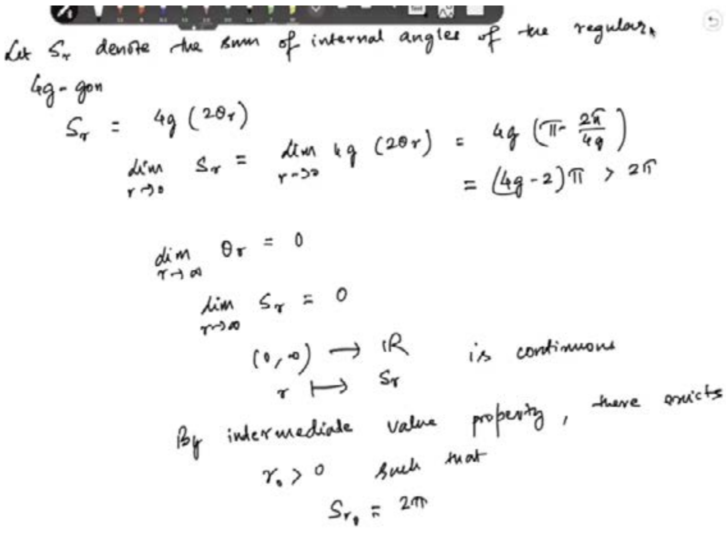

Let S_r represent the sum of the internal angles of a regular 4g-gon. Now, each internal angle is twice θ_r, and since the polygon has 4g sides, the sum of the internal angles, S_r, is $4g \times 2\theta_r$. Taking the limit as $r \to 0$, we get:

$$\lim_{r\to 0} S_r = \lim_{r\to 0} 4\,g \times 2\theta_r$$

From our previous result, we know:

$$\lim_{r\to 0} 2\,\theta_r = \pi - \frac{2\pi}{4g}$$

Substituting this into the equation for S_r, we have:

$$S_r = 4g \times \left(\pi - \frac{2\pi}{4g}\right)$$

Simplifying, we get:

$$S_r = 4g\pi - 2$$

Clearly, this is greater than 2π.

Next, consider the limit as $r \to \infty$. In this case, θ_r approaches zero because as r increases, the points P_r and Q_r move closer to the boundary of the disk, and the geodesics become asymptotically parallel. As $r \to \infty$, the internal angles θ_r tend to zero. Therefore, the sum of the internal angles in the limit becomes:

$$\lim_{r \to \infty} S_r = 0$$

Now, if we define a map from the interval (0, ∞) to R, where r is mapped to the sum of the internal angles, this map is continuous. From what we've established, as $r \to 0$, the sum of the internal angles is greater than 2π, and as $r \to \infty$, the sum approaches zero.

By the intermediate value theorem, there must exist some positive value r_0 such that $S_{r_0} = 2\pi$. Thus, we've proved the claim: there exists a regular 4g-gon with geodesic sides such that the sum of the internal angles is exactly 2π.

It's important to note that for g = 1, we cannot obtain a square in the unit disk with a hyperbolic metric where the internal angles are exactly $\pi/2$. This configuration simply doesn't exist in hyperbolic geometry.

In the next class, we will focus on performing side pairings in this 4g-gon, where $g \geq 2$. Specifically, we'll work with a genus 2 surface by considering a regular octagon. By making appropriate identifications along the boundary of this octagon, we can generate a genus 2 surface, which will allow us to construct a hyperbolic metric on it. I will stop here for now.