An Introduction to Hyperbolic Geometry

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Module - 7

Lecture – 23

Covering Spaces and Polygonal Constructions of Hyperbolic Surfaces

Hello everyone! In today's lecture, we will be revisiting the fascinating topic of covering space theory. We will explore how a surface can be expressed as a quotient of either R^2 or the upper half-plane through the lens of this theory. So, let us begin.

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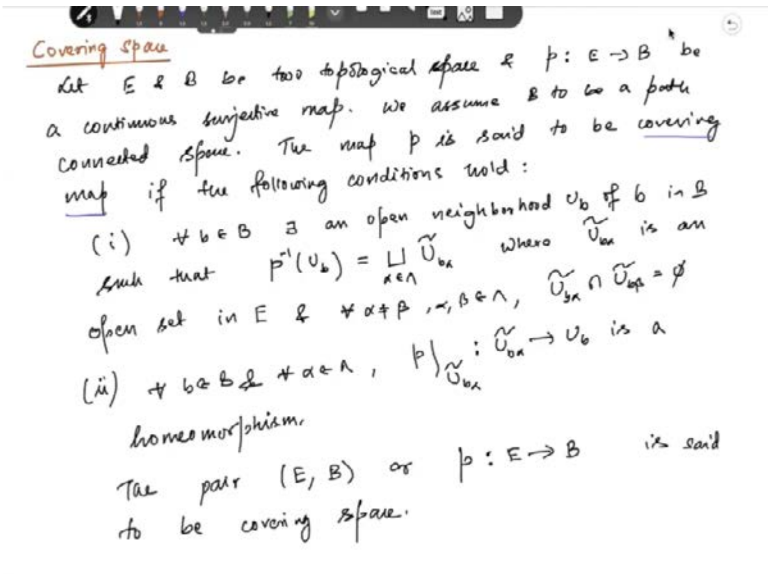

Let us begin by defining what a covering space is. Consider two topological spaces, denoted as E and B, and assume there exists a continuous surjective map $p: E \to B$. We will also assume that the space B is path-connected. The map p is termed a covering map if it satisfies the following conditions:

First, for every point b in B, there exists an open neighborhood U_b around b such that the preimage $p^{-1}(U_b)$ is a disjoint union of open sets in E. We denote these open sets as $U_b^{\tilde{\alpha}}$, where for any indices α and β in the index set λ, the sets $U_b^{\tilde{\alpha}}$ and $U_b^{\tilde{\beta}}$ are disjoint when $\alpha \neq \beta$.

The second condition states that if we restrict the map p to one of these open sets, say $U_b^{\tilde{\alpha}}$, then this restricted map is a homeomorphism. Specifically, for each b and α, the restriction $p|_{U_b^{\widetilde{\alpha}}}$ is a homeomorphism from $U_b^{\widetilde{\alpha}}$ to U_b.

With this definition in hand, the pair (E, B, p) is referred to as a covering space. Now, let us explore some examples to illustrate this concept further.

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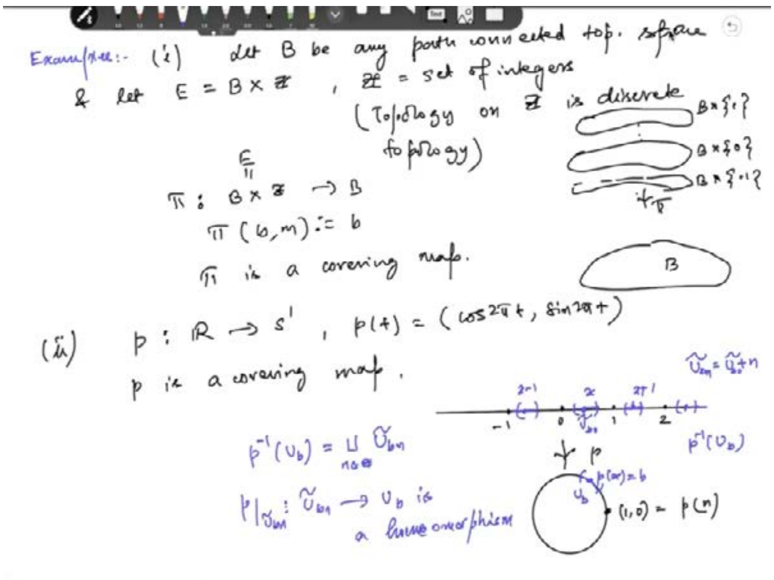

Let us begin with our first example. Consider a path-connected topological space B and define E as $B \times Z$, where Z represents the set of integers equipped with the discrete topology. We then have a projection map from $B \times Z$ to B. In this context, E is simply $B \times Z$.

Now, what does this projection map look like? If we take an element $(b, m) \in B \times Z$, the projection $\pi(b, m)$ simply yields b. This π is indeed a covering map, and the underlying structure is quite simple.

To visualize this, imagine that B is represented on the ground, while above it, we have $B \times 0, B \times -1$, and so forth, continuing indefinitely in both directions. This structure indicates that E is a disconnected topological space. Here, we refer to E as the total space and B as the base space. These terminologies are borrowed from the concept of fiber bundles.

Now, let's move on to our second example. Consider the real line R with the usual topology and the unit circle S^1 embedded in R^2. The topology on the circle comes from the subspace topology inherited from R^2. We define a map $p: R \to S^1$ given by:

$$p(t) = (\cos(2\pi t)\,, \sin(2\pi t)).$$

This map p is a covering map and is surjective. To understand this mapping, let us visualize R as a horizontal line with points labeled; for instance, we can mark 0 and 1 as our integer points. The function p maps this line onto the circle, with p(0) corresponding to the point (1, 0) on the circle.

Notably, any integer point p(n) for $n \in Z$ will also map to the point (1, 0). However, if we take any point x that is not an integer, say x, then p(x) will map it to some point on the circle. We can then select a small neighborhood around p(x), which we denote as U_b.

Now, what does $p^{-1}(U_b)$ yield? It gives us a collection of open intervals around x. After applying a translation by integers, we can visualize this as generating points x + n for all $n \in Z$. Specifically, $U_b^{\tilde{0}}$ would consist of intervals around x, while $U_b^{\tilde{n}}$ corresponds to these intervals translated by n. Thus, we obtain a disjoint union of these neighborhoods.

This means that $p^{-1}(U_b)$ is indeed a disjoint union of all the translated open sets $U_b^{\tilde{n}}$. Additionally, the way we have chosen $U_b^{\tilde{0}}$ ensures that the restriction of p is a homeomorphism. Therefore, we conclude that the map p is a covering map.

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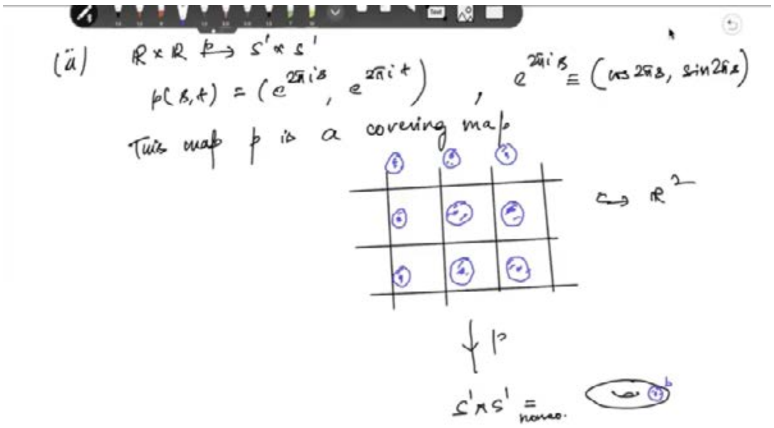

Continuing in the same philosophical vein, let us explore another example. We can consider the action defined on $R \times R$ mapping to $S^1 \times S^1$. The mapping p is specified by the equation:

$$p(s,t) = \left(e^{2\pi is}, e^{2\pi it}\right).$$

Here, $e^{2\pi is}$ is equivalent to the point represented by $(\cos(2\pi s), \sin(2\pi s))$. Once again, it is straightforward to verify that this map p is indeed a covering map.

To visualize this, let us consider a lattice positioned within R^2. From this lattice, we can define a map p from R^2 to $S^1 \times S^1$, which is homeomorphic to the torus. Upon inspection, we find that for any point b in $S^1 \times S^1$, the preimage $p^{-1}(b)$ consists of a discrete set of points. To understand this better, we begin by selecting a lift of the point b, which resides somewhere within the unit square.

For instance, if we take one lift of the point b, other lifts can be obtained by translating this point vertically by a length of 1 and horizontally by a length of 1, and this process can be continued indefinitely. Consequently, these lifted points of b are distributed throughout the plane.

Now, around the point b, we can establish a small open neighborhood such that the inverse image of that neighborhood would look like a collection of points resembling this. Additionally, we can consider another scenario: suppose we have a covering map p_i from E_i to B_i. If we take the Cartesian products $E_1 \times E_2$ and $B_1 \times B_2$, then $B_1 \times B_2$ will form a covering map from $E_1 \times E_2$ to $B_1 \times B_2$.

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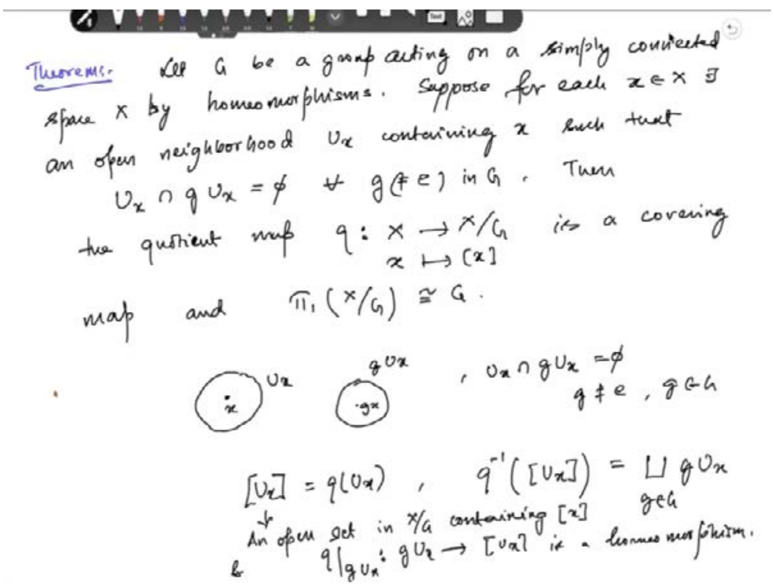

Now, let's shift our focus to group actions. The previous example involving the torus can also be derived from the action of $Z \times Z$ on R^2. Here, $Z \times Z$ operates through translations in R^2, leading us once again to the torus. Let’s revisit this theorem once more for clarity.

Let G be a group acting on a simply connected space X via homeomorphisms. For every element x in this topological space X, we can find an open set or neighborhood U_x that contains the element x. This neighborhood has a crucial property: the intersection of U_x with the set obtained by translating U_x by any group element g (where g ≠ the identity element) is empty. In other words, for all $g \in G$ except for the identity, we have:

$$U_x \cap gU_x = \emptyset.$$

Now, with this setup, we can define the quotient map q from X to the quotient space X / G, where X / G represents the set of orbits of the action of G on X. This quotient map is indeed a covering map. But why is that the case?

It's quite straightforward to understand. Around each point x in the base space X / G, we have the neighborhood U_x. When we translate this neighborhood by any non-identity element g, the translated neighborhoods remain disjoint. Thus, the quotient map identifies the point x with its image g x.

Let’s express this more formally. We denote the equivalence class of x under the action of G by [x]. The map q sends U_x to the corresponding equivalence class $[U_x]$. The preimage under the quotient map, $q^{-1}([U_x])$, can be seen as the disjoint union of all the sets g U_x, where g varies over the group G. This can be represented as:

$$q^{-1}([U_x]) = \bigsqcup_{g \in G} g\, U_x.$$

Each g U_x is an open set in X and contains the equivalence class of x. Additionally, the restriction of the map q from g U_x to the open set $[U_x]$ is a homeomorphism. Therefore, we conclude that the map q is indeed a covering map.

Now that we have established this foundational concept, let’s explore how we can construct a torus from this framework.

Let us again consider R^2 and examine the action of $Z \times Z$ on R^2 through translations. This action satisfies the conditions outlined in the theorem. For any point, or any element (m, n), we can find an open set $U_{m,n}$ that contains this point (m,n) such that the intersection with the translated points remains empty for all (m, n) not equal to (0, 0).

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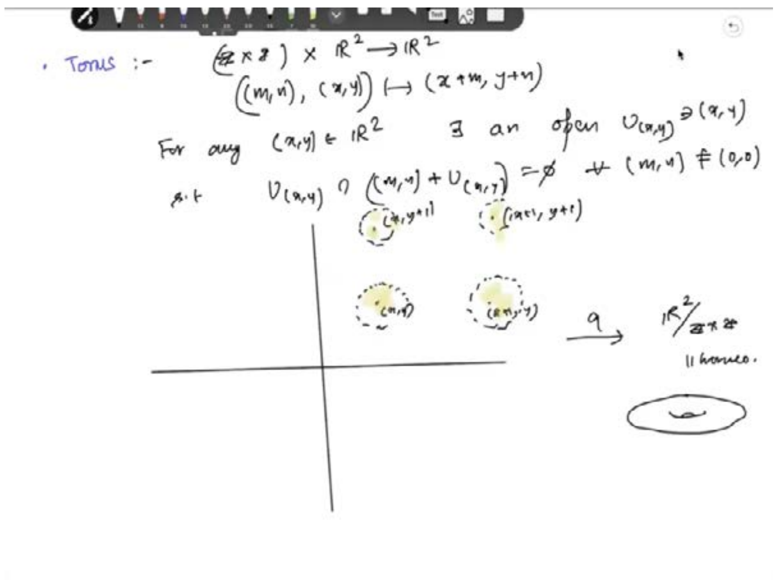

To visualize this, let's draw a picture. For any point (x, y) in R^2, there exists an open set $U_{x,y}$ containing the point (x,y) such that the intersection is empty for all (m, n) not equal to (0, 0).

If we take a specific point (x, y), we note that translating by (1, 0) would move us to (x+1, y), and translating by (0, 1) would bring us to (x, y+1). Now, if we consider a small open ball around the point (x, y) and translate this ball by (1, 0), we obtain a new ball that does not intersect with the ball centered at (x, y). This guarantees that we can always find such open sets that maintain this disjoint property.

The overall picture looks something like this, with the open balls clearly separated. When we take the quotient under this action, we obtain a map p whose quotient space is homeomorphic to a torus.

To further clarify why this quotient space is indeed homeomorphic to a torus, let me illustrate another perspective.

Let's take a closer look at the picture. First, we begin by establishing a lattice in R^2. Here, $Z \times Z$ acts on R^2, placing this lattice firmly within our R^2 plane. We can designate the horizontal axis as the x-axis. Now, observe that in the quotient space, the point (0, 0) is identified with (1, 0), and similarly, (1, 0) is identified with (2, 0). This means that any point of the form (n, 0) is effectively identified with the origin. However, it’s important to note that if I take two distinct points within this interval, they remain unconnected, they are not identified with one another.

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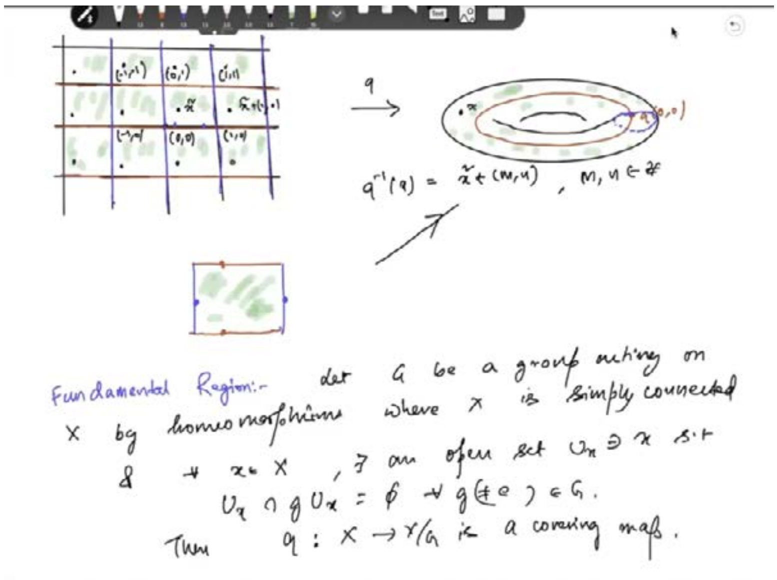

Now, if we consider the quotient of the red line, it will form a loop on the torus; let’s denote this point as q(0, 0). Moving on to the vertical direction, we can define the y-axis as this blue line, where the quotient of this line will also yield a loop. If we apply the translation (0, 1), we see that the red line shifts to the line defined by y = 1. Applying the quotient map q on this line again yields a closed curve on the torus.

Continuing with this reasoning, if we take all horizontal lines, their quotients will also correspond to this red loop. Conversely, if we consider all vertical lines, the quotient of these lines will differ merely by a translation. Hence, the quotient of all these vertical lines will correspond to the blue curve originating from the point q(0, 0).

Now, let’s focus on any point on the torus. The lift of that point will reside somewhere within this structure. For instance, if we consider one lift of this point, we can denote it as $\tilde{x}$, which lies within the unit square. In fact, the lift of this point will extend to points within every unit square surrounding it. For example, you can think of this point as $\tilde{x} + (1,0)$; these represent the various lifts of the point x. Therefore, the inverse image $q^{-1}(x)$ is expressed as $\tilde{x} + (m, n)$, where (m, n) are integers.

Importantly, if we restrict our attention to the interior of any unit square, we find that all these interiors remain disjoint from one another. Ultimately, what we obtain is a square, where the sides are identified but the interiors remain distinct. So, here, one corner is identified with another, and likewise for the opposite corners. This identification, as we've discussed earlier, leads to the formation of a torus, with the sides correlating to those red and blue curves.

This interior region we've been discussing is referred to as the fundamental region. Let me define it formally. The fundamental region arises in the following context: Let G be a group acting on X via homeomorphisms, where X is simply connected. For every point x in the space X, there exists an open set U_x that contains x such that the intersection $U_x \cap gU_x$ is empty for all g not equal to the identity element. Under these circumstances, the quotient map from X to X mod G acts as a covering map.

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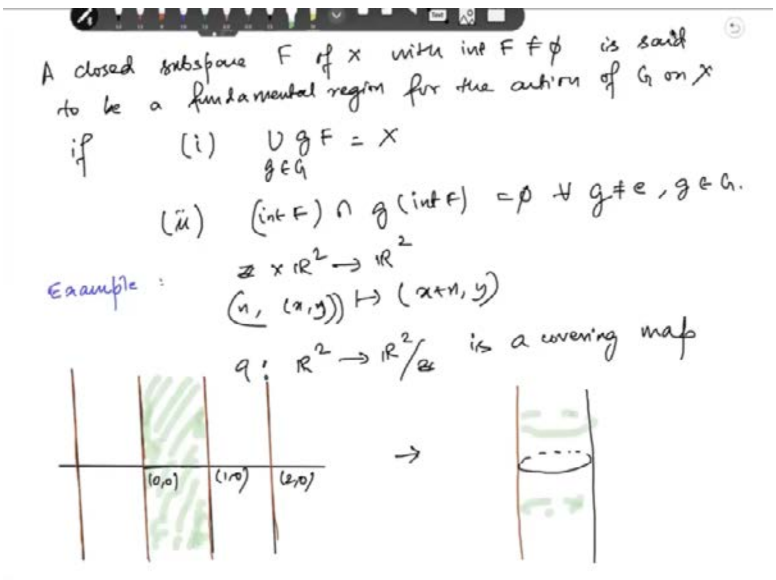

A closed subspace F of X is deemed a fundamental region for the action of G on X if its interior is non-empty and it satisfies two primary conditions. The first condition requires that the union of all translates of F equals the entirety of X. Formally, this means that

$$\bigcup_{g \in G} gF = X.$$

The second condition stipulates that the intersection of the interior of F with any translate of the interior must be empty:

$$\operatorname{int}(F) \cap g\operatorname{int}(F) = \emptyset \quad \text{for all } g \neq e.$$

To illustrate this concept, consider the example of a unit square derived from the lattice; this unit square serves as a fundamental domain.

Now, let’s examine another example involving the action of Z^2. We can refer to this as an "oriental action." This action, as established by the theorem, fulfills the required conditions, which allows us to assert that the quotient map $q: R^2 \to R^2/Z^2$ functions as a covering map.

Next, let's identify the fundamental region in this scenario. Suppose we designate a point as the origin, with the point (1, 0) located directly to its right. Here, we can define our fundamental region by taking a subspace of R^2 that encompasses both boundaries; this will form our fundamental region.

Now, what does the quotient look like in this case? The quotient is represented as a cylinder, where all the red vertical lines in our visualization are identified as a single line. Thus, when we consider this interval, it transforms into a closed loop, ultimately giving rise to a cylinder.

Now, if we want to extend this discussion to a genus 2 surface, one can proceed with the following construction.

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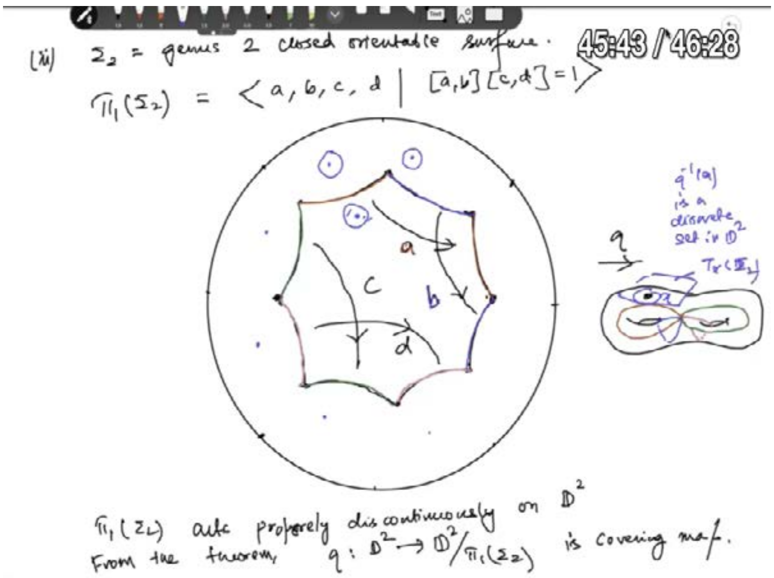

The genus 2 closed orientable surface, denoted as Σ_2, has a fundamental group generated by four elements: a, b, c, d. The relations among these elements can be expressed through the product of commutators, which equals 1. Now, let us consider the Poincaré disk model. This concept will be demonstrated in a future class. If we take the unit disk and equip it with the hyperbolic metric, we can inscribe a regular octagon within this model of the hyperbolic plane.

In this setting, we can define an isometry, denoted as e, which maps one geodesic to another. Additionally, we can have another isometry, referred to as b, that transforms one blue geodesic to another. Similarly, there exists an isometry c that maps one geodesic to a different geodesic, and finally, an isometry d that transitions one geodesic to yet another. By performing these side pairings on the octagon, we arrive at a quotient that produces a genus 2 surface.

In this quotient space, the red geodesic corresponds to a closed loop associated with the blue geodesic, while the pink and green geodesics similarly yield closed loops. It can be demonstrated that the fundamental group of the genus 2 surface acts properly discontinuously, thereby satisfying the conditions outlined in our theorem.

In some literature, this action is referred to as a covering space action or a properly discontinuous action. We can indeed prove that the fundamental group $\pi_1(\Sigma_2)$ acts properly discontinuously on the unit disk model of the hyperbolic plane. From the theorem, we can conclude that the map in question is a covering map, which is crucial for our discussion.

This leads us to an important conclusion: any closed orientable surface of genus 2 can admit a hyperbolic structure. In other words, there exists a covering map q that implies the existence of a hyperbolic metric on the genus 2 surface. With this hyperbolic metric established on the unit disk, we can push this metric onto the surface of genus 2 through the covering map.

Now, if we take any point in this setting, the lift of that point will yield an 8×8 tiling of the unit disk. Visualizing this tiling can be quite challenging. For any point x, the inverse image $q^{-1}(x)$ is a discrete set within the unit disk. At this point x, we can find an open set such that its inverse image appears in a particular manner. Furthermore, by considering the tangent plane at this location, we can transfer the hyperbolic inner product from the unit disk to this tangent plane. Thus, this covering map facilitates the transfer of the hyperbolic metric from the unit disk to the genus 2 surface. I will conclude here.