An Introduction to Hyperbolic Geometry

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Module - 7

Lecture – 22

Polygonal Constructions of Hyperbolic Surfaces: From Tori to Higher Genus

Hello and welcome to this lecture! Today, we will explore the fascinating process of constructing surfaces from polygons. For example, if we take a unit square and identify its opposite sides, we will create a torus. Similarly, if we take a regular octagon and appropriately identify its sides, we will end up with a closed orientable surface of genus 2. This concept opens up a world of possibilities in the study of topology!

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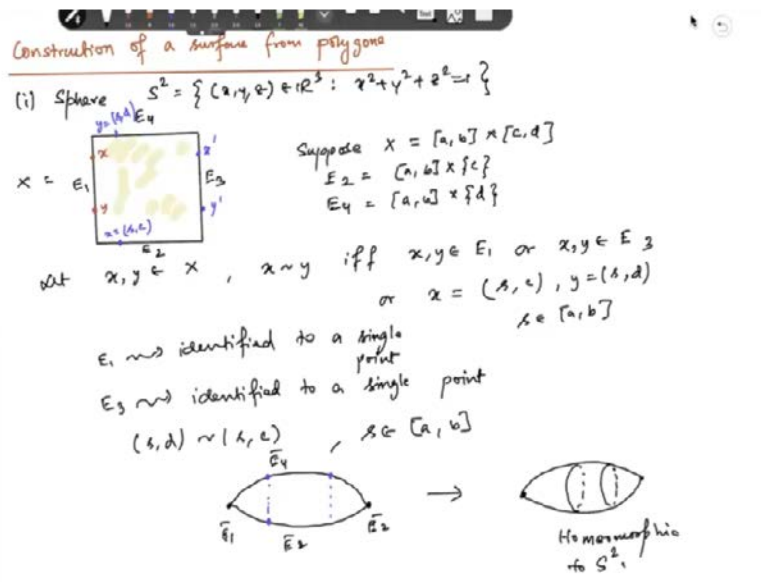

Let's begin by constructing a sphere from a square. First, we'll define what we mean by a sphere in this context. Specifically, we can consider the unit sphere in R^3. Now, let's take a unit square, denoted as X.

To relate this square to a sphere, we need to establish an equivalence relation on this set, which will allow us to create a quotient space that is homeomorphic to the unit sphere.

So, what's the underlying idea here? We will take any two points x and y in the square X. Let's denote the boundary of this square, where X is comprised of both the interior and the edges. For clarity, we'll label the sides of the square as E_1, E_2, E_3, and E_4.

Now, the relation is defined such that points x and y are considered equivalent if they both belong to either E_1 or E_3. This means that if x is on side E_1 and y is also on side E_1, they will be identified with each other. Likewise, if points x' and y' are located on E_3, they will also be identified.

Additionally, there is another crucial identification to consider. Suppose we represent the square as the interval $[a, b] \times [c, d]$. Here, E_2 corresponds to the edge defined by $[a, b] \times \{c\}$, and E_4 corresponds to $[a, b] \times \{d\}$. By establishing these relationships, we can successfully construct a surface that corresponds to a sphere.

Now, let's clarify the relationship between the points x and y. Initially, we established that both x and y can belong to either E_1 or E_3, or alternatively, x can equal (s, c) and y can equal (s, d), where s lies within the interval [a, b]. In this setup, the point (s, c) is identified with the point (s, d).

The key assertion here is that the resulting quotient space is homeomorphic to a sphere. It's important to note that the left edge of the square or rectangle, E_1, is identified as a single point. Similarly, the edge E_3 is also condensed into a single point.

If we were to make only these identifications, the quotient space would still be homeomorphic to the rectangle or square. However, with the additional identification where (s, d) is identified with (s, c) as s varies over the closed interval [a, b], the resulting figure will look like this: when we identify E_1 as a point, it gives us a singular point, and when E_3 is similarly identified, it also results in a point.

Let's denote these points as E_1' and E_2'. If we were to consider only this identification, the configuration would resemble this: the side E_4' would take a specific shape, and this would represent our new edge E_2'.

Now, we proceed to identify this point with that point, and this other point with yet another point. Once we implement these changes, the next figure we obtain will resemble something like this, and it will indeed be homeomorphic to a sphere.

To illustrate this more concretely, let's write down the explicit homeomorphism that achieves this transformation.

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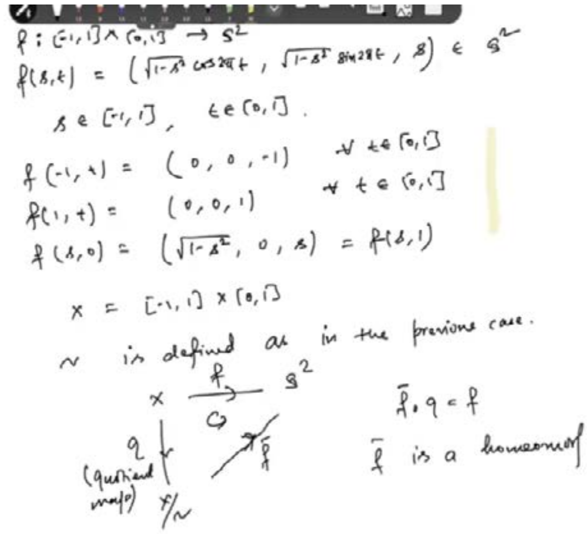

Let’s define the mapping f(s, t) as follows:

$$f(s,t) = \left(\sqrt{1-s^2}\cos(2\pi t), \sqrt{1-s^2}\sin(2\pi t), s\right).$$

Here, the first coordinate corresponds to $\sqrt{1-s^2}\cos(2\pi t)$, while the second coordinate is $\sqrt{1-s^2}\sin(2\pi t)$. The point (f(s, t)) thus lies on the unit sphere, with s varying from -1 to 1 and t varying from 0 to 1. Consequently, we see that f maps the rectangle defined by the interval $[-1,1] \times [0,1]$ onto the sphere S^2.

Now, consider specific cases: when we evaluate f(-1, t), we find that it yields the point (0, 0, -1) for all t within the unit interval. Similarly, evaluating f(1, t) gives us the point (0, 0, 1) for all t in the same interval.

Next, if we examine f(s, 0), we see that the first coordinate results in $\sqrt{1-s^2}$, the second coordinate remains 0, and the third coordinate is simply s. Notably, this is equivalent to f(s, 1).

Now, let’s denote the space X as our rectangle, and we observe that the relation among these points is defined similarly to our previous discussion. As we proceed, we visualize a diagram where there is a map from X to the sphere S^2, accompanied by a quotient map from X to a quotient space, which we’ll denote as Y.

This quotient space corresponds precisely to the structure we discussed, and due to these three equalities we've identified, the function f will induce a map from the quotient space Y to S^2. Thus, we can assert the existence of a map $\overline{f}$ from this quotient space to S^2, such that the following diagram commutes:

$$\overline{f} \circ q = f,$$

where q denotes the quotient map. It can also be proven that $\overline{f}$ is indeed a homeomorphism. So, this is a comprehensive overview of our mapping onto the sphere.

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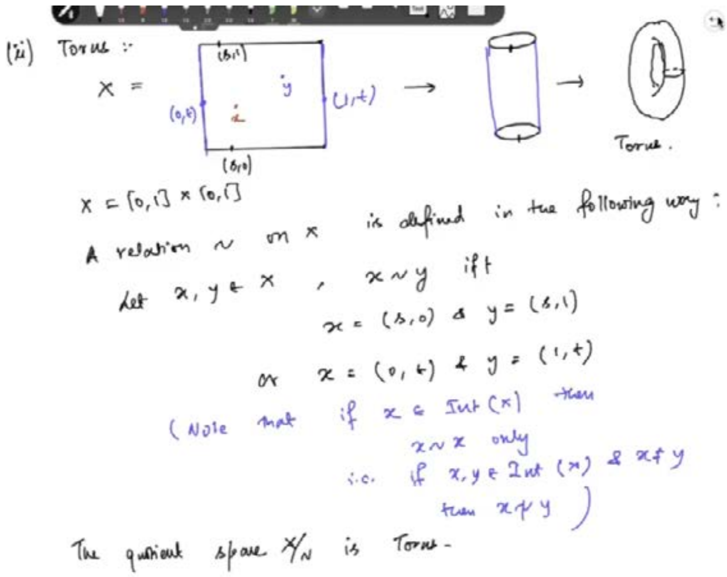

Now, let's turn our attention to the construction of a torus. We can start with a square, although a rectangle will work just as well. Imagine this square in front of us. The identification process begins here. Let's define our space X as the unit interval crossed with itself, which essentially forms a square.

Now, consider how we identify points within this space. We define a relation on X in the following manner: take two points x and y belonging to X. We say that x is related to y if and only if x takes the form (s, 0) and y takes the form (s, 1). This means that any point on the bottom edge of the square is identified with its corresponding point on the top edge.

Alternatively, we can also identify points where x is of the form (0, t) and y is of the form (1, t). In this case, any point on the left edge is identified with its corresponding point on the right edge. This identification occurs solely on the boundary of the square.

It’s crucial to note that if a point x lies in the interior of X (that is, it does not touch the boundary), then it is only related to itself. In other words, if we have two distinct points x and y both within the interior, they are not related. Thus, we only have identifications occurring along the boundary of our square.

Now, with this identification process in mind, let’s visualize what happens when we apply these relations. When we first identify the top and bottom edges of the square, those edges transform into circles, effectively forming a cylinder. The process of identification connects these edges in a way that the top edge becomes one circle and the bottom edge becomes another, leading us to the shape of a cylinder.

Now, as we continue with our identification process, we notice that one point is identified with another. Specifically, this corresponds to the pairs (s, 0) and (s, 1). This brings us to yet another identification. To recap, the first identification involved the bottom and top edges, while the second pertains to the left and right edges. Through these identification processes, we successfully construct a torus.

What we have established here is that the quotient space obtained from our unit square indeed forms a torus. It’s quite fascinating because we can explicitly describe a mapping from this quotient space to the torus itself.

Now, let's visualize the torus as a surface of revolution. Imagine taking a unit circle situated in the y-z plane, positioned at some positive distance away from the z-axis so that it does not pass through the origin. When we rotate this circle around the z-axis, we generate a beautiful surface of revolution, which is our torus.

Furthermore, we can express the parametric equations of this torus derived from the unit square. If you delve into the details, you will find that this mapping provides a homeomorphism from our quotient space to that surface of revolution. Hence, we can confidently conclude that this quotient space is homomorphic to the torus.

Now, let us shift our focus to higher genus surfaces. Remember, the torus is classified as a genus 1 surface, and exploring surfaces of higher genus will reveal even more intriguing topological properties.

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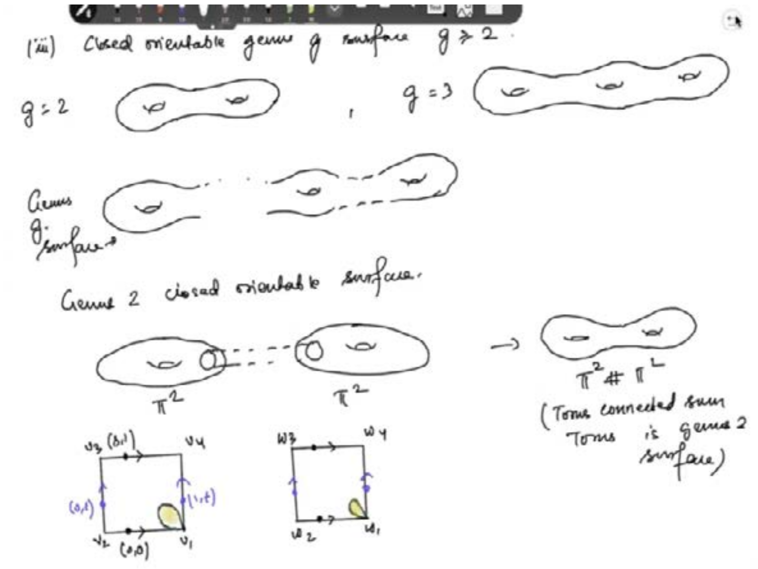

Now, let’s discuss the closed orientable surfaces of genus g, specifically when g is greater than or equal to 2. What do these surfaces look like? For instance, if we set g = 2, we obtain a genus 2 surface, and if g = 3, we have a genus 3 surface, as illustrated in our diagram. As we move to even higher genus surfaces, we can similarly represent them through diagrams.

To construct a genus 2 closed orientable surface, we begin with two copies of a torus. The first step involves removing a closed disk from each of these tori, which creates a hole in each. Next, we glue these two tori together along the resulting circular edges, ultimately forming the genus 2 surface. This process of connecting two surfaces by removing disks and then gluing them together is known as the connected sum operation. Hence, the connected sum of two tori gives us the genus 2 surface.

Now, let's revisit the process of obtaining a torus. We start with a square. In this case, we have two copies of the torus represented as squares. We identify the opposite sides of each square: for example, one point on one square is identified with a corresponding point on the opposite edge, while the same goes for the other square. To help visualize this identification process, we can use arrows to indicate the direction of identification.

For instance, if we say that the point (s, 0) is identified with (s, 1), the arrow signifies this correspondence. Similarly, if we have the point (0, t) identified with (1, t), the arrow indicates that they are to be glued together.

Once we have these two copies of the torus ready, our next step is to remove a disk from each torus. We can visualize this by considering a loop at each corner of the torus. Each of these loops will enclose a disk. Therefore, by removing these disks from both tori, we set ourselves up for the final construction of our genus 2 surface.

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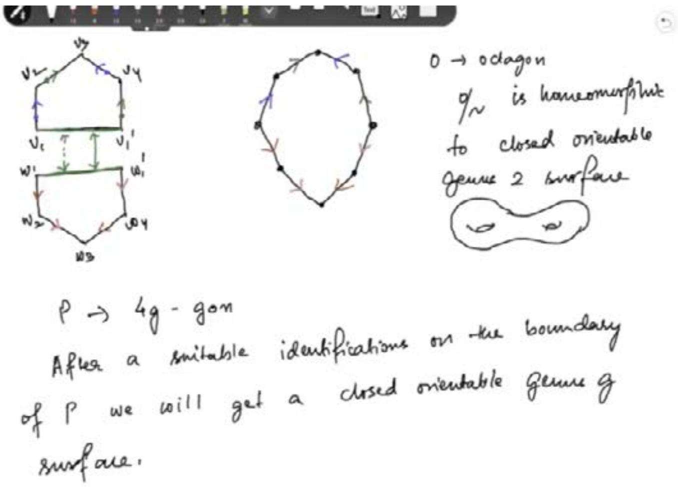

Now, let's visualize what happens when we remove the disks from the two tori: we are left with two pentagons. In this representation, let’s label the vertices of these pentagons as v_1, v_2, v_3, and v_4. After removing the disk from the first torus, we’ll specifically refer to the vertices at the ends of the edges as v_1 and v_1'. So we have the points v_1, v_1' along with v_2, v_3, and v_4.

It's crucial to note that if we identify v_1 with v_1', we establish a direct relationship between these two points. Similarly, we can make identifications for the sides. If we identify the vertices v_2 and v_3, this creates another crucial relationship. By continuing this process, if we identify v_1 with v_1' once again, we effectively return to our original configuration.

Now, let’s turn our attention to the second torus. We will perform the same operation, which yields another pentagon. To distinguish this, let's label the vertices of the second pentagon as w_1, w_2, w_3, and w_4, along with w_1'. At this stage, we now have two pentagons, each with its own set of identifications. For example, one side of the first pentagon is identified with the corresponding side of the second pentagon.

What we aim to achieve here is to glue these two pentagons together. The yellow regions we removed from both squares represent where these shapes will connect. By attaching along the edges defined by these loops, we form the connected sum of the two tori, creating our genus 2 surface.

To clarify this process further, let's consider that the loop around the first pentagon corresponds to a specific line, while in the second pentagon, the loop corresponds to the line joining w_1 and w_1'. As we proceed, we will identify these two green lines, effectively gluing them together. This action brings us closer to forming the desired structure of the genus 2 surface.

We arrive at an octagon through this process. Specifically, we can consider it to be a regular octagon, since our discussion is focused on homeomorphism, which allows for such a consideration. So, we will have a regular octagon represented here. Now, let's examine the identification process.

In this context, one side of the octagon is identified with its opposite side, and this pattern continues: the second side is identified with the corresponding side, the third side is paired with its opposite, and finally, the fourth side is matched with its corresponding side.

After completing this identification, we will denote the octagon as O, noting that our identifications occur solely on the boundaries. This relationship allows us to define the equivalence relation among the sides. The result of these identifications leads us to a quotient space that is homeomorphic to a closed orientable genus 2 surface, as illustrated by the figure presented here.

Now, we can generalize this construction. If we consider a polygon with 4g sides, denoted as P, after applying suitable identifications to the boundary of P, we will arrive at a closed orientable surface of genus g.

I will conclude my explanation here by emphasizing that the construction of the genus 2 surface, as well as those of higher genus surfaces, highlights an important property: any closed orientable surface with a genus greater than or equal to 2 admits a hyperbolic metric. This means that at each point on the surface, we can define a tangent space, and on that tangent space, we can consistently apply the hyperbolic metric. So, I will stop here.