An Introduction to Hyperbolic Geometry

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Module - 7

Lecture – 21

Free Groups and Group Presentations in Hyperbolic Geometry

Hello. In this lecture, we will learn about free groups and presentation of a group. So, we will prove that if you take any group, it is a quotient of a free group. So, let us begin.

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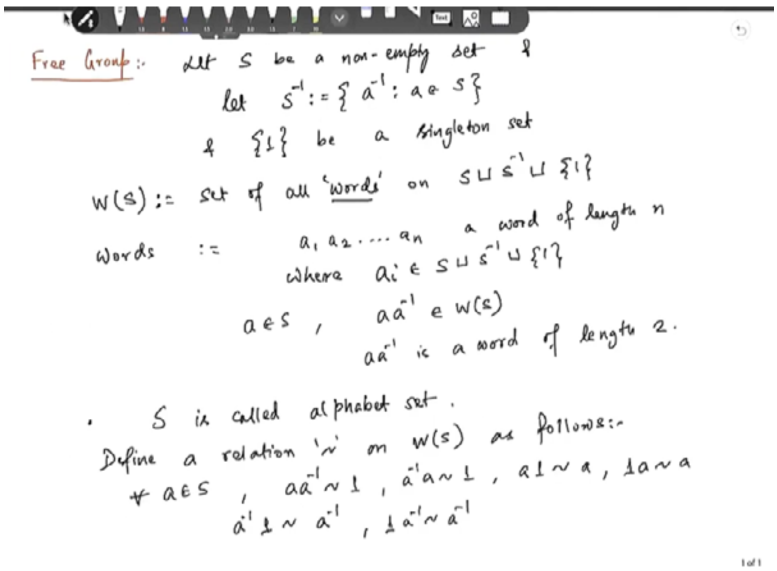

Let's start by defining what we mean by a free group. Consider a non-empty set S. We will also define another set, S^{-1}, which consists of the symbols a^{-1} for each element a that belongs to S. It's essential to understand that these symbols are just that, symbols without any inherent meaning. Additionally, we will include a singleton set, denoted as {1}, where 1 should also be treated as a symbol.

Now, our goal is to define a free group generated by S. To do this, we first need to clarify what we mean by the set of all words formed from S. We denote this set as W(S). Specifically, W(S) is defined as the disjoint union of S, S^{-1}, and the singleton set {1}.

When I refer to "words," I mean any element of the form $a_1 a_2 \ldots a_n$, which is a word of length n where each a_i belongs to either S, S^{-1}, or {1}.

For instance, if we take some a belonging to S, then the expression a a^{-1} is a word of length 2. Similarly, a^{-1} a also represents a word of length 2. At this point, we can refer to S as the "alphabet set."

From this set of all words, we want to construct a group, which is our free group. The first step in this construction is to define a relation on the set of all words. Here's how we do it:

For every a belonging to S, we declare that a a^{-1} is related to 1, and likewise, a^{-1} a is also related to 1. This means that in the quotient group, both a a^{-1} and a^{-1} a are equivalent to the identity element 1. Additionally, if a_1 is a word in S, we relate it to a since, at the group level, we want $a \cdot 1 = a$. Thus, this relation holds true.

Continuing this process, we also state that 1 a is related to a, a^{-1} 1 is related to a^{-1}, and 1 a^{-1} is related to a^{-1}. It's important to note that this definition only applies to the elements in S.

Finally, we can extend this relation to encompass the entire set W(S).

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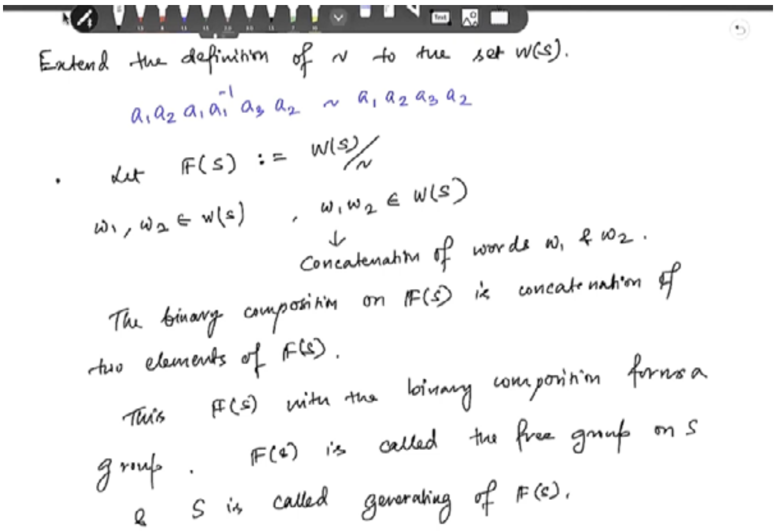

Let's now extend the definition of this relation to the set W(S). For instance, consider the word a_1 a_2 a_1 a_1^{-1} a_3 a_2. This word can be simplified because a_1 a_1^{-1} is identified with 1, which means we can eliminate it. Consequently, the expression becomes equivalent to a_2 a_3 a_2. Thus, we have established a relation within W(S).

Now, we can define the quotient group by considering the set of all equivalence classes, which constitutes our quotient space. This means that if we take S as a set, we treat it as a collection of these equivalence classes.

Next, we need to introduce a binary operation on this free group F(S) that will provide it with a group structure. Let's take two words, denoted as w_1 and w_2, from F(S). Notably, both w_1 and w_2 also belong to W(S). When we place w_1 and w_2 side by side, we create a new word, which again is an element of W(S). Therefore, we have $w_1 w_2$, which is still a word formed from S, S^{-1}, and {1}.

This operation, which we will refer to as the concatenation of the words w_1 and w_2, serves as our binary composition. It provides a straightforward way to combine two elements of F(S).

Thus, the binary composition defined by the concatenation of two elements from F(S) allows us to conclude that F(S), equipped with this binary operation, indeed forms a group. The identity element of this group is simply 1, making this structure a free group on S. Therefore, we refer to F(S) as the free group generated by S, and we call the set S the generating set of this free group.

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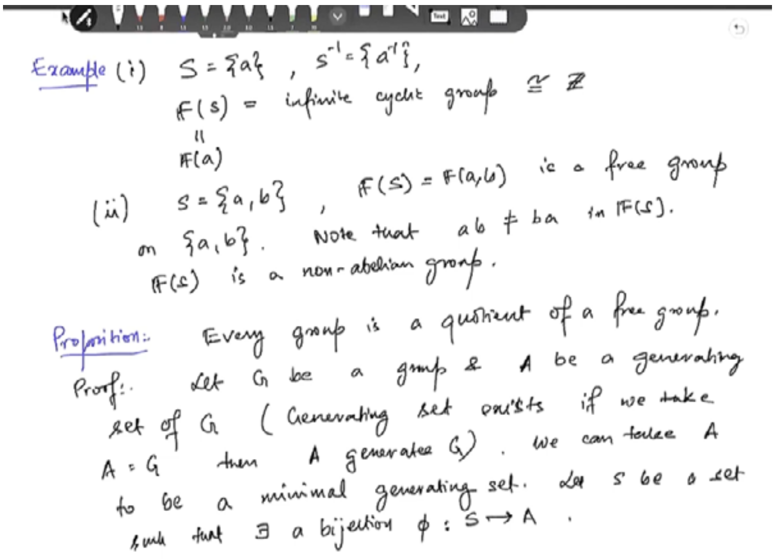

Let's consider an example: suppose we take the set S to be a singleton set {a}. In this case, the inverse set S^{-1} consists of the symbol a^{-1}. Consequently, the free group generated by this set S is nothing but an infinite cyclic group, which is isomorphic to the set of integers Z. Thus, we can denote this free group as F(S) when S is a singleton set, and it can also be expressed as F(a).

Now, let’s move on to a different scenario where S is a two-point set, say {a, b}. In this case, we denote the free group as F(S), or more explicitly as F(a, b), indicating that it is a free group generated by the two symbols a and b.

It's important to note that this group is not cyclic. In fact, within this free group, the word ab is distinct from the word ba. This difference proves that the free group generated by two elements is a non-abelian group.

Now, let’s consider the next proposition, which states that every group is a quotient of a free group. How do we prove this?

We start by letting G be any group and define A as a generating set of G. The existence of this generating set is guaranteed because, if we take A = G, then A clearly generates G. Hence, we can always find such a generating set. Additionally, we can select A to be a minimal generating set, if necessary.

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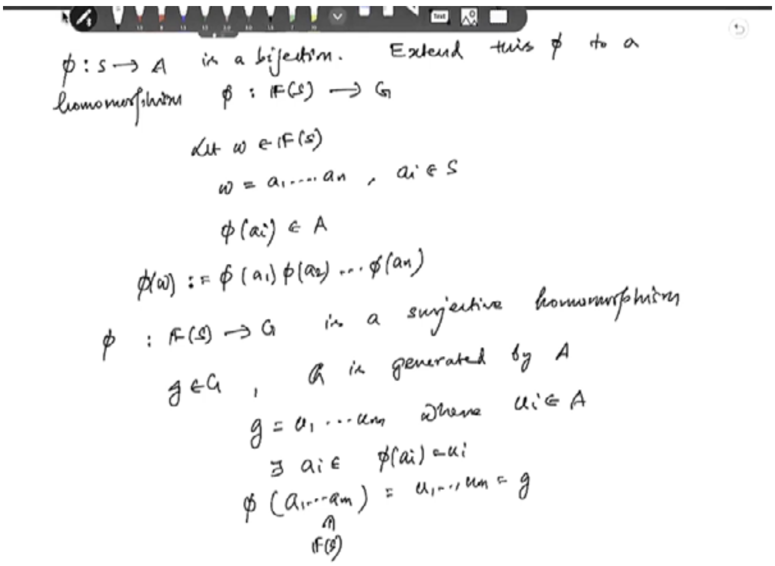

Next, we introduce another set S. We can define S such that there exists a bijection φ from S to A. This is always possible; whenever you have a set A, you can find another set S with the same cardinality as A. Thus, we can create a proper correspondence between the two sets, which will be useful in our proof.

Now that we have established a bijection from the set S to the set A, we can extend this bijection to create a homomorphism from the free group F(S) to the group G. I will denote this extended map by the same symbol φ, that is, $\phi: F(S) \to G$.

Let's explore how this extension works. Consider an arbitrary element w belonging to the free group F(S). We can express w in the form $a_1 a_2 \dots a_n$, where each a_i belongs to S. According to our bijection, each a_i corresponds to an element in A.

Thus, we can define φ(w) by taking $\phi(a_1), \phi(a_2), \dots, \phi(a_n)$. Since each $\phi(a_i)$ maps to an element of G, we can conclude that φ(w) will yield an element in the group G as well. Therefore, we have successfully extended the bijection to a homomorphism from the free group F(S) to the group G.

Moreover, this map φ from the free group to the group G is a surjective homomorphism. This property is straightforward to verify.

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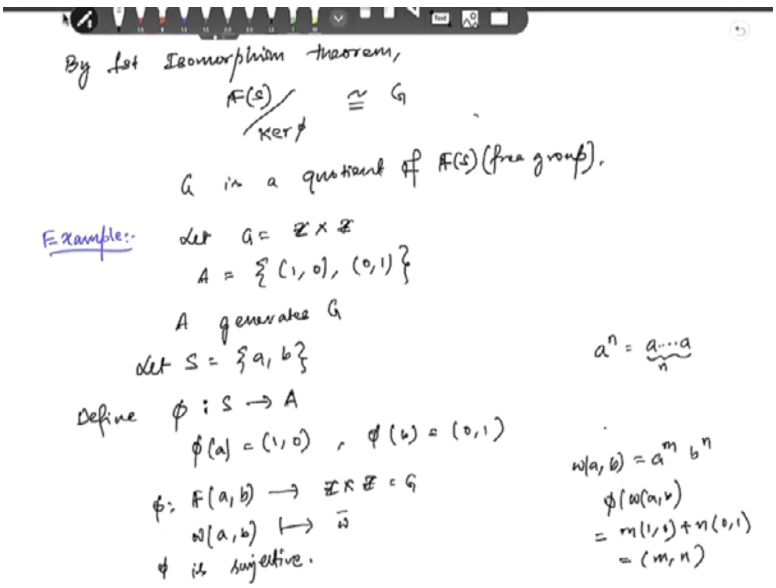

To see why, consider any element g in the group G. Since G is generated by the set A, we can express g as a product of elements $u_1, u_2, \dots, u_m$, where each u_i belongs to A. Given that φ is a bijection from S to A, there exists some a_i such that $\phi(a_i) = u_i$.

Now, if we take the word $a_1 a_2 \dots a_m$ in the free group F(S) and apply φ, we find that $\phi(a_1 a_2 \dots a_m) = u_1 u_2 \dots u_m$, which indeed belongs to G. Thus, we have shown that this homomorphism is surjective, confirming that every element in G can be reached through φ.

Now, if we apply the First Isomorphism Theorem, we find that the quotient group formed by the free group F(S) divided by the kernel of φ, which is a normal subgroup, is isomorphic to the group G. This result conclusively demonstrates that G is indeed a quotient of the free group F(S).

With this foundation laid, let's explore an example to solidify our understanding. Consider the group G defined as $Z \times Z$. Our goal is to identify a free group such that G serves as a quotient of that free group.

We know that if we take the set A to consist of the vectors (1, 0) and (0, 1), these vectors generate G. Next, let us define the set S to be a two-point set consisting of elements a and b. We can then establish a bijection $\varphi: S \ \rightarrow A$ as follows: we define φ(a) = (1, 0) and φ(b) = (0, 1). This gives us a bijection between S and A, which can be extended from the free group generated by these two symbols to the group generated by A, which is $Z \times Z$.

Now, let's clarify what this homomorphism φ entails. Suppose we take a word w in the letters a and b, such as $w = a^m b^n$. Here's what happens under our homomorphism: the image of a is (1, 0), and a^m represents the concatenation of the letter a, m times. Similarly, b^n corresponds to n concatenations of b.

Therefore, when we apply φ to $w = a^m b^n$, we obtain:

$$\phi(w) = \phi(a^m)\phi(b^n) = m \cdot (1,0) + n \cdot (0,1) = (m, n).$$

As a result, corresponding to the word w in the free group F(S), we derive an element $\overline{w}$ in $Z \times Z$.

Moreover, we can affirm that the map φ is surjective, ensuring that every element in G can be achieved through φ. This example effectively illustrates the relationship between free groups and quotient groups in the context of group theory.

According to the First Isomorphism Theorem, we can conclude that this quotient group is isomorphic to $Z \times Z$, which is precisely our group G. Now, let's move on to the next claim: we need to determine the kernel of the homomorphism φ. The kernel of φ is identified as the normal closure of the word $aba^{-1}b^{-1}$, which we denote as such.

But what does the term "normal closure" of the word $aba^{-1}b^{-1}$ mean? It refers to the smallest normal subgroup that contains the word $aba^{-1}b^{-1}$. Now, let's delve into how we can prove this.

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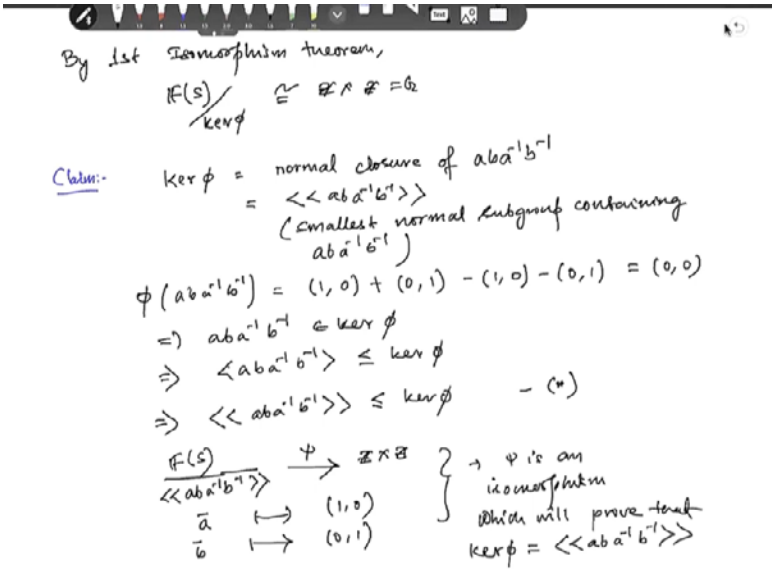

First, note that when we evaluate $\phi(aba^{-1}b^{-1})$, we find that it equals (1, 0) + (0, 1). Specifically, $\varphi(a^{-1})$ is equal to (-1, 0), so when we compute (-1, 0) + (0, -1), we arrive at (0, 0). This indicates that the word $aba^{-1}b^{-1}$ indeed belongs to the kernel of φ. Thus, we have established that the subgroup generated by $aba^{-1}b^{-1}$ is a subgroup of the kernel of φ.

Moreover, we recognize that if we take any conjugate of the word $aba^{-1}b^{-1}$, it will also belong to the kernel of φ. Therefore, this normal closure is also a subgroup of the kernel of φ.

In summary, we have proven that the normal closure is indeed a subgroup of the kernel of φ. Now, let's consider the quotient group. If we take this group and define a map, let's call it ψ, for an element $\bar{a}$ in the quotient group, we set $\psi(\bar{a}) = (1,0)\ and\ \psi(\bar{b}) = (0,1)$.

What we have here is a homomorphism, and in fact, this is an isomorphism. This result confirms that the kernel of φ is precisely equal to the normal closure containing the element $aba^{-1}b^{-1}$. Thus, we have achieved a significant milestone in our exploration of the relationships between these groups!

Let's discuss a crucial concept in group theory: the commutator. The expression $aba^{-1}b^{-1}$ is known as the commutator, and we denote it using the notation [a, b].

Now, I would like to introduce the notion of the presentation of a group. Consider a group G. We can express G as a quotient of a free group, which we denote as F(S). In this context, G is understood to be a free group that has been factored out by some normal subgroup. This normal subgroup is specifically the normal closure of a subset R within the free group F(S).

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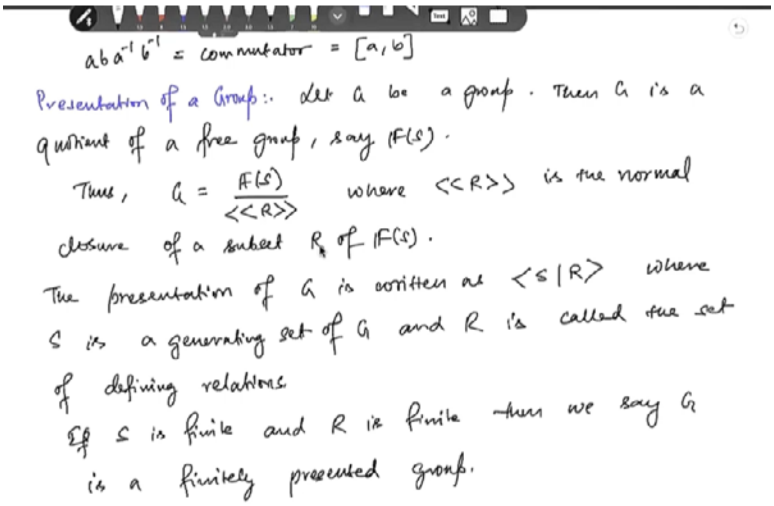

The presentation of G is conventionally written in the following manner:

$$G = \langle S \mid R \rangle$$

Here, S represents the generating set of G, while R is referred to as the set of defining relations. It is worth noting that if both S and R are finite sets, we categorize G as a finitely presented group.

Next, I want to briefly define two additional concepts: the free product of groups and the amalgamated product of groups. These concepts are fundamental in understanding how groups can be constructed and related to one another.

Let's consider two groups, denoted as G_1 and G_2. Each of these groups has its own presentation. Specifically, let G_i represent the presentation for i = 1 and 2. Now, I want to introduce the concept of the free product of groups.

From the groups G_1 and G_2, we will construct a new group G. Let’s define a set S as the disjoint union of S_1 and S_2, and let R be the disjoint union of R_1 and R_2. We will then consider the group G to be generated by this set S, with the relations given by R.

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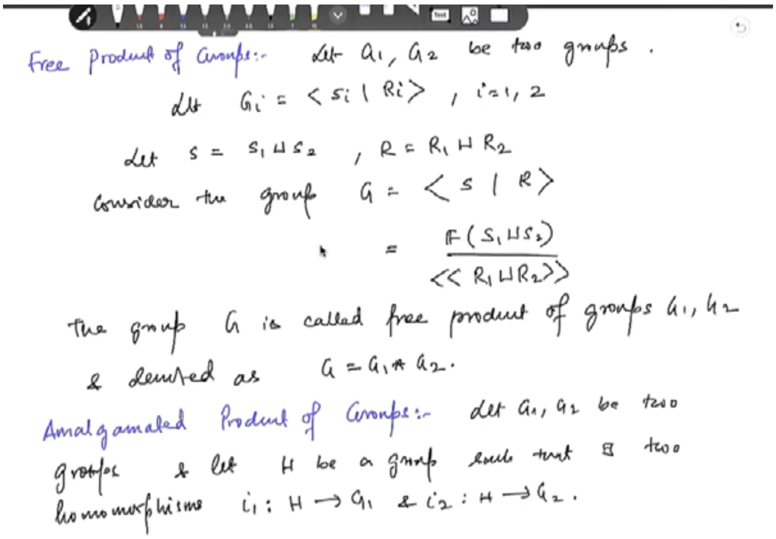

Mathematically, this means that G can be represented as the free group on the set S_1 disjoint union S_2 with the normal closure of the relations $R_1 \cup R_2$. Therefore, we denote the group G as the free product of the groups G_1 and G_2, which we write as:

$$G = G_1 * G_2$$

This notation indicates that $G_1 * G_2$ is the free product of the groups G_1 and G_2.

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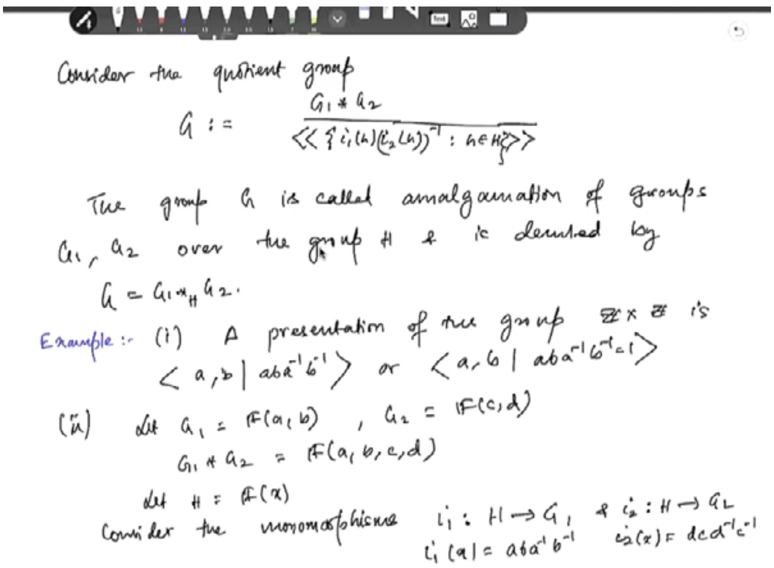

Next, let’s define the amalgamated product of groups. Again, consider the two groups G_1 and G_2, along with a third group H. In this case, we assume there exist two homomorphisms: $i_1: H \to G_1$ and $i_2: H \to G_2$. Our goal now is to construct another group from these three groups: G_1, G_2, and H.

Now, let's define the quotient group G. The group G is constructed as follows: we begin with the free product $G_1 * G_2$, which is the free product of G_1 and G_2, and then we take the normal closure of a specified set within this free product $G_1 * G_2$. The resulting group G is referred to as the amalgamation of the groups G_1 and G_2 over the group H, and it is denoted by:

$$G = G_1 *_H G_2$$

It's important to note that if we take H to be a trivial subgroup, then this amalgamation simplifies to a standard free product.

Now, let’s look at an example to illustrate this concept. The presentation of the group $Z \times Z$ can be expressed in terms of its generators and relations. Specifically, the group is generated by the elements a and b, with the defining relation given by $aba^{-1}b^{-1} = 1$. We can also represent this relation as ab = ba. This commutativity is why we state that $aba^{-1}b^{-1} = 1$.

For the next example, let’s define G_1 as the free group generated by a and b, and G_2 as the free group generated by c and d. The free product $G_1 * G_2$ is simply the free group generated by the elements a, b, c, and d.

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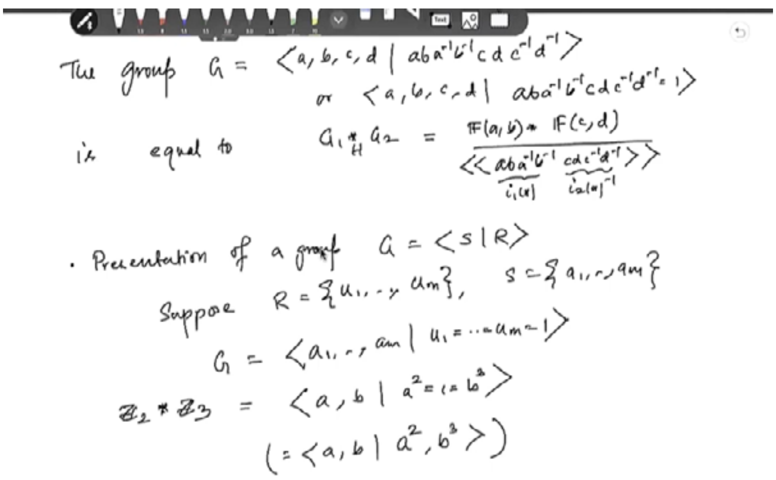

Now, let's consider H to be a free group generated by a single element x. This means that H is an infinite cyclic group generated by x. We will define the monomorphisms $i_1: H \to G_1$ and $i_2: H \to G_2$. Specifically, i_1 is defined by mapping x to $aba^{-1}b^{-1}$, and i_2 is defined by mapping x to $dcd^{-1}c^{-1}$. With these definitions in place, we can now consider the constructed group.

The group G is generated by the elements a, b, c, and d, with the defining relation expressed as $aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1$. To clarify this notation, we can state that G is generated by a, b, c, and d, and the relation signifies that the element $aba^{-1}b^{-1}cdc^{-1}d^{-1}$ is equal to the identity element.

This group G is effectively the amalgamation of G_1 and G_2 over the group H. In this context, G_1 represents the free group generated by two elements, a and b, while G_2 is the free group generated by the elements c and d. The amalgamation process involves taking the free product of G_1 and G_2 and then quotienting by the normal subgroup generated by the relation $aba^{-1}b^{-1}$ and the relations involving c and d.

So, what have we learned so far? Any group can be represented as a quotient of a free group. When we refer to the presentation of a group, we mean that G is generated by a set S, with a corresponding set of defining relations R. For instance, if we denote the relations as $u_1, \ldots, u_m$ and the generating set as $a_1, \ldots, a_n$, we can write the presentation of G as follows:

$$G = \langle\, a_1, \ldots, a_m \mid u_1 = u_2 = \cdots = u_m = 1 \,\rangle$$

This notation provides another way to represent the group G.

As a practical example, consider the free product $Z_2 * Z_3$. This group is indeed the free product of Z_2 and Z_3, which is generated by two elements, say a and b. Here, a has an order of 2, while b has an order of 3. The defining relations can be expressed as $a^2 = 1$ and $b^3 = 1$.

Furthermore, we can adopt another notation for this presentation, stating that the group is generated by a and b, and we can succinctly write it as:

$$\langle\, a, b \mid a^2, b^3 \,\rangle$$

This illustrates the same group structure in a slightly different notation, capturing the essence of the generators and their respective relations.