

# An Introduction to Hyperbolic Geometry

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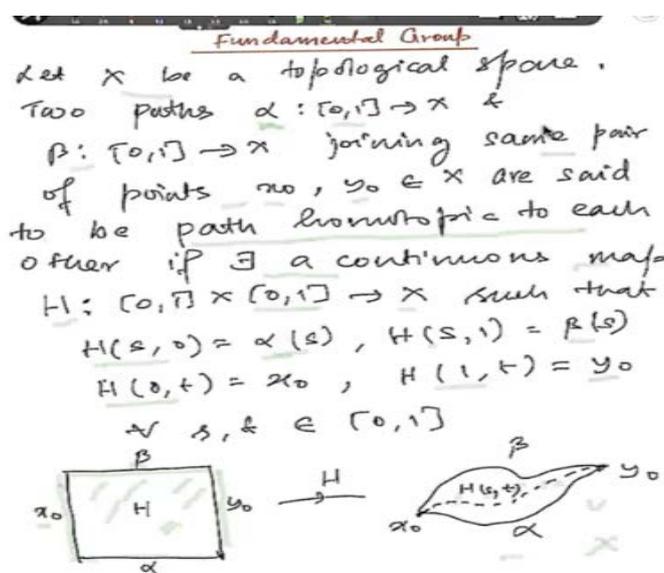
Indian Institute of Technology – Kanpur

Lecture: 20

## Fundamental Group

Hello, and welcome to this lecture! Today, we will delve into the fascinating concept of the fundamental group. Understanding the fundamental group is essential, as it will play a crucial role in our exploration of non-elementary Fuchsian groups. Our primary objective in this lecture is to demonstrate that the fundamental group of a closed orientable surface with genus greater than or equal to 2 corresponds to Fuchsian groups, and indeed, these groups are non-elementary Fuchsian groups. So let us begin.

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To grasp the concept of the fundamental group, we first need to understand what is meant by path homotopy. Let's start by considering a topological space  $X$  and two paths within it. But what exactly is a path? A path is simply a continuous map from the closed interval  $[0, 1]$  into  $X$ .

Now, let's denote two such paths as  $\alpha$  and  $\beta$ . Both of these are continuous maps from the closed interval  $[0, 1]$  to  $X$  and connect the same pair of points, which we will call  $x_0$  and  $y_0$  in  $X$ .

These two paths,  $\alpha$  and  $\beta$ , are said to be path homotopic to one another if there exists a continuous map  $H$  from the unit square  $[0,1] \times [0,1]$  to  $X$ . More precisely, we require that:

- $H(s, 0) = \alpha(s)$
- $H(s, 1) = \beta(s)$

for each  $s \in [0,1]$ . Furthermore, we must have the conditions:

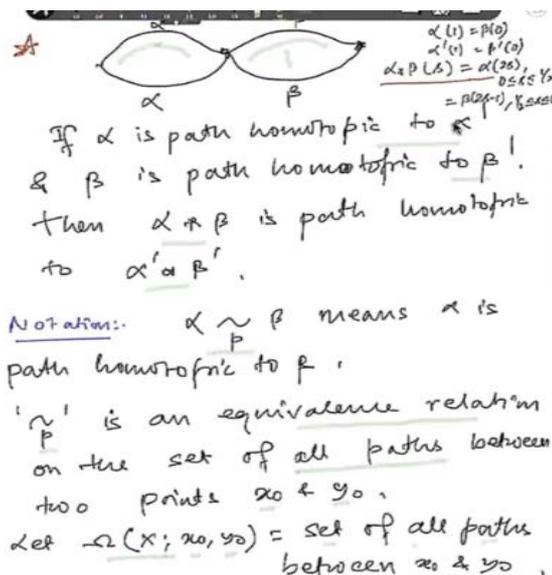
- $H(0, t) = x_0$  for all  $t$
- $H(1, t) = y_0$  for all  $t$

Here, the second closed interval  $[0, 1]$  can be interpreted as a time parameter. At time  $t = 0$ , the map  $H(s, 0)$  corresponds to the path  $\alpha$ , while at time  $t = 1$ ,  $H(s, 1)$  corresponds to the path  $\beta$ .

For every time  $t$ ,  $H(0, t)$  remains fixed at  $x_0$ , and at the opposite end,  $H(1, t)$  remains fixed at  $y_0$ .

If we visualize this, we can create a schematic diagram to illustrate the homotopy. In this diagram, the unit square represents the space over which  $H$  is defined. At the bottom edge, where  $H(s, 0)$  lies, we have the path  $\alpha$ , while at the top edge, where  $H(s, 1)$  is located, we find the path  $\beta$ .

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On the left side of the square,  $H(0, t)$  is consistently equal to  $x_0$ , and on the right side,  $H(1, t)$  is consistently equal to  $y_0$ . Thus, if we take the image of the path inside the square, it maps to a path in the space  $X$  that illustrates the homotopy from  $\alpha$  to  $\beta$ . This entire construction is referred

to as path homotopy.

Next, we can demonstrate that if path  $\alpha$  is homotopic to path  $\alpha'$  and path  $\beta$  is homotopic to path  $\beta'$ , we can consider their concatenation. Specifically, let's denote the concatenation of  $\alpha$  and  $\beta$  as  $\alpha * \beta$ , where the endpoint of  $\alpha$  coincides with the initial point of  $\beta$ , and similarly, the terminal point of  $\alpha'$  matches the initial point of  $\beta'$ . For clarity, we will assume that  $\alpha(1) = \beta(0)$  and  $\alpha'(1) = \beta'(0)$ .

Now, what does this concatenation signify? This concept is likely familiar from your algebraic topology class. The concatenation  $\alpha * \beta$  is defined as follows:

$$(\alpha * \beta)(s) = \begin{cases} \alpha(2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ \beta(2s - 1) & \text{if } \frac{1}{2} < s \leq 1 \end{cases}$$

Similarly, you can define the concatenation  $\alpha' * \beta'$ .

Now, let's consider the result: if  $\alpha$  is homotopic to  $\alpha'$  and  $\beta$  is homotopic to  $\beta'$ , we can prove that  $\alpha * \beta$  is path homotopic to  $\alpha' * \beta'$ . This notation means that  $\alpha$  is related to  $\beta$  if  $\alpha$  is path homotopic to  $\beta$ .

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$\Omega(X; x_0, y_0) / \sim_p$   
 $[\alpha] =$  equivalence class containing  $\alpha$ .  
 $[\alpha] * [\beta] \stackrel{\text{defn.}}{=} [\alpha * \beta]$   
 (Binary composition on  $\Omega(X; x_0, y_0) / \sim_p$ )  
 Note that this definition is well defined.  
Associativity:  $\alpha, \beta, \gamma \in \Omega(X; x_0, y_0)$   
 Then  $\alpha * (\beta * \gamma)$  is path homotopic to  $(\alpha * \beta) * \gamma$   
Proof:  $(\alpha * (\beta * \gamma))(s) = \alpha(2s), 0 \leq s \leq \frac{1}{2}$   
 $= (\alpha * \beta)(2s-1), \frac{1}{2} < s \leq 1$

Moreover, one can verify that this relationship is indeed an equivalence relation on the set of all paths between the two points  $x_0$  and  $y_0$ . Consequently, if we denote the set  $\Omega(X, x_0, y_0)$ , it

encompasses all paths between  $x_0$  and  $y_0$ .

I have established this relation on the set, and it indeed forms an equivalence relation. In this context, the notation  $[\alpha]$  denotes the equivalence class containing the path  $\alpha$ . Due to this property, we can verify that if we take two paths,  $\alpha$  and  $\beta$ , such that  $\alpha(1) = \beta(0)$ , the concatenation  $\alpha * \beta$  is well-defined, thus providing a binary operation on the set  $\Omega(X, x_0, y_0)$ .

This binary operation can be extended into the quotient space. In this framework,  $[\alpha] * [\beta]$  signifies the equivalence class of the concatenation  $\alpha * \beta$ . Importantly, this composition is well-defined, ensuring that it makes sense in our context. Additionally, we possess the property of associativity.

Now, let's consider three paths:  $\alpha$ ,  $\beta$ , and  $\gamma$ , each connecting  $x_0$  to  $y_0$ . The first step is to compute  $\beta * \gamma$  and then concatenate  $\alpha$  with this result. Alternatively, we can first concatenate  $\alpha$  with  $\beta$  and subsequently concatenate this result with  $\gamma$ .

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Handwritten mathematical derivations and a diagram illustrating the composition of paths  $\alpha$ ,  $\beta$ , and  $\gamma$ .

Diagram labels:  $(0,0)$ ,  $(1,0)$ ,  $(1,1)$ ,  $(0,1)$ ,  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\alpha * \beta$ ,  $(\alpha * \beta) * \gamma$ .

Equations:

$$\begin{aligned} &= \beta(4s-2), \quad \frac{1}{2} \leq s \leq \frac{3}{4} \\ &= \gamma(4s-3), \quad \frac{3}{4} \leq s \leq 1. \\ (\alpha * \beta) * \gamma(s) &= (\alpha * \beta)(2s), \quad 0 \leq s \leq \frac{1}{2} \\ &= \alpha(2s-1), \quad \frac{1}{2} \leq s \leq 1 \\ (\alpha * \beta) * \gamma(s) &= \alpha(4s), \quad 0 \leq s \leq \frac{1}{4} \\ &= \beta(4s-1), \quad \frac{1}{4} \leq s \leq \frac{1}{2} \\ &= \gamma(2s-1), \quad \frac{1}{2} \leq s \leq 1. \\ &\alpha = (\beta * \gamma) \circ (0,1) \times \\ &(\alpha * \beta) * \gamma = (0,1) \rightarrow \alpha \\ &= \text{Image}(\alpha * (\beta * \gamma)) \\ &= \text{Image}(\alpha * (\beta * \gamma)) \end{aligned}$$

$$\begin{aligned} H(s,t) &= \alpha\left(\frac{4s}{2-t}\right), \quad 0 \leq s \leq \frac{2-t}{4} \\ &= \beta(4s-t-2), \quad \frac{2-t}{4} \leq s \leq \frac{2-t}{4} \\ &= \gamma\left(\frac{4s+t-3}{1-t}\right), \quad \frac{3-t}{4} \leq s \leq 1 \end{aligned}$$

Remarkably, both approaches yield the same element in the quotient space. Hence, we define the concatenation of these paths as:

$$\alpha * \beta * \gamma = \begin{cases} \alpha(2s) & \text{for } 0 \leq s \leq \frac{1}{2} \\ (\beta * \gamma)(2s - 1) & \text{for } \frac{1}{2} < s \leq 1 \end{cases}$$

This formulation clearly captures the essence of the operations we are performing on these paths.

If we expand the expression, we find that  $\alpha * \beta * \gamma$  can be represented as follows:  $\alpha(2s)$  for  $s$  varying from  $0$  to  $\frac{1}{2}$ . Next, considering  $\beta * \gamma$ , we can express it in this manner:  $\beta(4s - 2)$  for  $s$  varying from  $\frac{1}{2}$  to  $\frac{3}{4}$ , and then as  $\gamma(4s - 3)$  for  $s$  varying from  $\frac{3}{4}$  to  $1$ . This leads us to a corresponding expression for  $\alpha * \beta * \gamma$ .

Now, observe that we have two paths: one is  $\alpha * \beta * \gamma$  and the other is also  $\alpha * \beta * \gamma$ . Both of these paths map from the closed interval  $[0, 1]$  to  $X$ , and they are continuous maps. Although the images of these two maps are identical, the only distinction lies in their parameterizations. Consequently, since these two parameterizations are continuous, they are path homotopic to each other.

So, what does this homotopy look like? To visualize it, consider a schematic diagram where you have the unit square. At the bottom of the square, you will have the path defined by  $\alpha * \beta * \gamma$ , and at the top, you have the path defined by  $\alpha * \beta * \gamma$ . To establish the homotopies, draw a straight line joining the point  $(\frac{1}{2}, 0)$  to  $(\frac{1}{4}, 1)$ , and another line connecting the point  $(\frac{3}{4}, 0)$  to  $(\frac{1}{2}, 1)$ .

Next, take this horizontal line segment into consideration. The intersection point's coordinates will be given by  $(\frac{2-t}{4}, t)$ , where  $s$  varies from  $0$  to  $\frac{2-t}{4}$ . Consequently, the term  $\frac{4s}{2-t}$  lies within the closed interval  $[0, 1]$ , allowing us to define  $\alpha$  along that segment as  $\alpha\left(\frac{4s}{2-t}\right)$ .

Similarly, for the other part, we can define  $\beta$  using its respective parameters, and for the next segment, we can use the equation of the line segment to define  $\gamma$  along this horizontal line segment.

Thus, we have established this function  $H$  to represent the homotopy. You can verify that at time  $t = 0$ ,  $H(s, 0)$  yields the path  $\alpha * \beta * \gamma$ , while at time  $t = 1$ ,  $H(s, 1)$  results in  $\alpha * \beta$  followed by the concatenation with  $\gamma$ .

Thus, we can conclude that these two elements are identical in the quotient space. Our primary objective here is to define the fundamental group. To establish a group, we require an identity element, and to produce this identity element, we will utilize the constant map. Let's denote this

constant map as  $C_{y_0}$ , which takes the value  $y_0$ . Specifically, we have  $C_{y_0}(s) = y_0$  for all  $s$  belonging to the closed interval  $[0, 1]$ .

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$([\alpha] * [C_{y_0}]) * [\gamma] = [\alpha] * ([\beta] * [\gamma])$

\* Let  $C_{y_0}: [0, 1] \rightarrow X$  denote the constant map taking value  $y_0$   
 i.e.  $C_{y_0}(s) = y_0 \quad \forall s \in [0, 1]$   
 Then  $\alpha * C_{y_0} \sim \alpha$

Proof:-

$H(s, t) = \alpha\left(\frac{2s}{t+1}\right), \quad 0 \leq s \leq \frac{t+1}{2}$   
 $= y_0, \quad \frac{t+1}{2} \leq s \leq 1$

\*  $[\alpha] * [C_{y_0}] = [\alpha]$   
 $[C_{y_0}] * [\alpha] = [\alpha]$

Now, the claim is that  $\alpha * C_{y_0}$  is path homotopic to  $\alpha$ . Here,  $\alpha$  represents a path, and  $C_{y_0}$  is a constant path. The concatenation  $\alpha * C_{y_0}$  is therefore well-defined, as  $\alpha$  is a path from  $x_0$  to  $y_0$ , making the combination  $\alpha * C_{y_0}$  valid. Furthermore, the image of this path is the same as the image of  $\alpha$ ; the only difference lies in the parameterization.

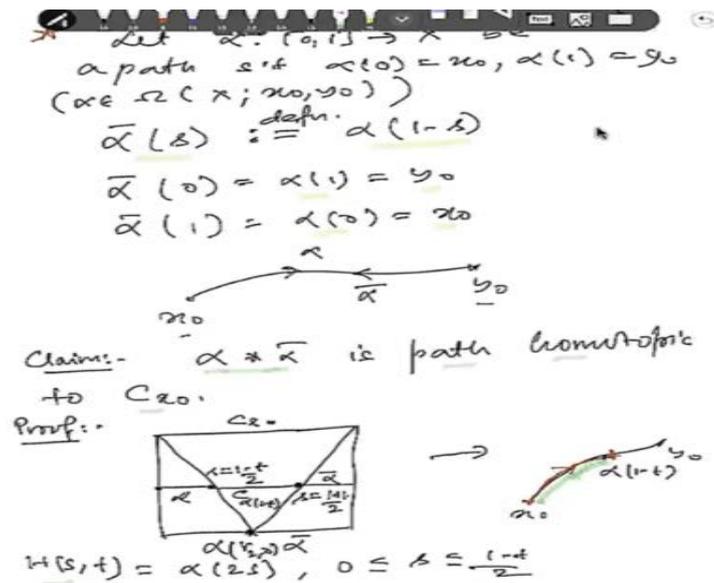
So how do we establish the homotopy? To visualize this, we take the unit square and connect the point  $(\frac{1}{2}, 0)$  to the point  $(1, 1)$  by a straight line. The equation of this straight line can be expressed as  $s = t + \frac{1}{2}$ . Next, we consider the horizontal line segment where  $s$  varies from 0 to  $t + \frac{1}{2}$ . In this interval, we define  $H$  as  $\alpha(2s/(t+1))$ , while the remainder of the path is defined as  $y_0$ . Hence, we obtain the image of  $\alpha$ .

Consequently, we have  $H(s, t) = y_0$  when  $s$  varies from  $(t + \frac{1}{2})$  to 1. This construction provides us with a homotopy between  $\alpha * C_{y_0}$  and  $\alpha$ .

Thus, in the quotient space, we can assert that the equivalence class of  $\alpha$  combined with the equivalence class of  $C_{y_0}$  is equal to the equivalence class of  $\alpha$ . Similarly, if we consider the constant map  $C_{x_0}$ , we find that the equivalence class of  $C_{x_0} * \alpha$ , is equal to the equivalence class

of  $\alpha$  as well.

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Now, consider any path  $\alpha$ . We can define  $\bar{\alpha}$ , which serves as the inverse of  $\alpha$  in the fundamental group. Specifically,  $\bar{\alpha}(s)$  is defined as  $\alpha(1 - s)$ . This means that  $\bar{\alpha}(0) = \alpha(1) = y_0$  and  $\bar{\alpha}(1) = \alpha(0) = x_0$ . We can visualize it this way: while  $\alpha$  starts at  $x_0$  and ends at  $y_0$ ,  $\bar{\alpha}$  effectively reverses this path, starting at  $y_0$  and concluding at  $x_0$ .

Next, we can prove that the concatenation  $\alpha * \bar{\alpha}$  forms a loop based at  $x_0$ ; it begins and ends at the same point,  $x_0$ . The assertion is that this path is homotopic to the constant map  $C_{x_0}$ . The proof is quite straightforward. The essential idea is that as we traverse the path, we move up to  $\alpha(1 - t)$  and then return back along  $\bar{\alpha}$ .

This process will yield the desired homotopy for any time  $t$ . To express this explicitly, we can define the homotopy  $H(s, t)$  as follows:  $H(s, t) = \alpha(2s)$  for  $s$  varying from  $0$  to  $\frac{1-t}{2}$ . This representation captures the transition along the path and demonstrates the homotopic relationship between  $\alpha * \bar{\alpha}$  and the constant map  $C_{x_0}$ .

Now, let's consider  $H(s, t)$  as the constant map. So what exactly is this constant map? It is defined as  $\alpha(1 - t)$ . In this interval,  $H(s, t)$  remains constant, equal to  $\alpha(1 - t)$ . When  $s$  varies from  $\frac{1+t}{2}$  to  $1$ , we define it as  $\bar{\alpha}(2s - 1)$ . This  $H$  provides the necessary homotopy.

Thus, the homotopy  $\bar{\alpha} * \alpha$  is path homotopic to the constant map  $C_{x_0}$ . This  $H$  clearly establishes

the homotopy, demonstrating that  $\alpha * \bar{\alpha}$  is homotopic to  $C_{x_0}$ . Similarly, we can conclude that  $\bar{\alpha} * \alpha$  is path homotopic to  $C_{y_0}$ . This leads us to the two important equations we have established.

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$H(s, t) = C_{\alpha(t)} = \alpha(1-t)$   
 $\frac{1-t}{2} \leq s \leq \frac{1+t}{2}$   
 $= \bar{\alpha}(2s-1), \frac{1-t}{2} \leq s \leq 1$   
 $\bar{\alpha} * \alpha \sim_p C_{y_0}$   
 $* [\alpha] * [\bar{\alpha}] = [C_{x_0}]$   
 $[\bar{\alpha}] * [\alpha] = [C_{y_0}]$   
Fundamental Group :-  
 Let  $X$  be a topological space  
 &  $x_0 \in X$   
 $\Omega(X, x_0) :=$  set of all paths (loops  
 at  $x_0$ )  $\alpha : [0, 1] \rightarrow X$   
 s.t.  $\alpha(0) = \alpha(1) = x_0$   
 $\pi_1(X, x_0) := \frac{\Omega(X, x_0)}{\sim_p}$

Now, let's move on to the definition of the fundamental group. Let  $X$  be a topological space, and let's fix a point  $x_0$  belonging to  $X$ . We define  $\Omega(X, x_0)$  as the set of all paths or loops at  $x_0$ , where each loop  $\alpha$  satisfies  $\alpha(0) = \alpha(1) = x_0$ .

But what do we mean by a loop? A loop is essentially a continuous map from the closed interval  $[0, 1]$  into  $X$ , where the endpoints remain the same, specifically at the point  $x_0$ . Now, the definition of  $\pi_1(X, x_0)$  comes into play. It is represented as the quotient space of  $\Omega(X, x_0)$ , encompassing the equivalence classes of all loops. Thus,  $\pi_1(X, x_0)$  captures the essence of the fundamental group for our topological space  $X$  based at the point  $x_0$ .

Now, considering everything we've discussed earlier, when we set  $y_0$  equal to  $x_0$ , we find that  $\bar{\alpha}$  becomes the inverse of  $\alpha$ . Consequently, the equivalence class of the constant map  $C_{x_0}$  acts as the identity element. This leads us to the conclusion that  $\pi_1(X, x_0)$ , with respect to this composition, indeed forms a group, which we refer to as the fundamental group of  $X$  based at the point  $x_0$ .

Next, let's examine a specific case: if we take  $X$  to be  $\mathbb{R}^n$ , we note that  $\mathbb{R}^n$  is a convex space. For any point  $x_0$  and another point  $x$  in this space, we can connect them with a straight line. If we define the homotopy  $H(x, t) = (1 - t)x + tx_0$ , this represents what we call the straight-line

homotopy. This straightforward relationship allows us to prove that the fundamental group of  $\mathbb{R}^n$  is trivial.

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$(\pi_1(X, x_0), *)$  forms a group and it is called fundamental group of  $X$  based at the point  $x_0$ .

Simply connected space: A path connected space  $X$  is said to be simply connected if its fundamental group  $\pi_1(X, x_0)$  is trivial.

Theorem (Armstrong): Let  $G$  be group acting on a simply connected space  $X$  by homeomorphisms. Suppose for each  $x \in X$  there exists an open set  $U_x$  in  $X$  s.t.  $x \in U_x$  &  $g \cdot U_x \cap U_x = \emptyset$   $\forall g \neq e \in G$ . Then  $\pi_1(X/G, [x_0]) \cong G$ , where  $x_0 \in X$ .  $X/G = \{ \bar{x} : x \in X \}$   $\bar{x} := \{ g \cdot x : g \in G \}$ .

*Handwritten notes:*  $G \times X \rightarrow X$   $(g, x) \mapsto g \cdot x$  is usual.

Now, let's clarify the definition of a simply connected space. A path-connected space  $X$  is deemed simply connected if its fundamental group  $\pi_1(X)$  is trivial. In other words, we begin with a path-connected space, and if we find that the fundamental group is trivial, we label that space as simply connected.

It's essential to note a fundamental fact: if we consider a path-connected space  $X$  and select two points  $x_0$  and  $x_0'$ , the fundamental group of  $X$  at  $x_0$  is isomorphic to the fundamental group of  $X$  at  $x_0'$ . This result is a standard outcome in basic algebraic topology.

Now, let's look at an example of a simply connected space. Since we have established that the fundamental group of  $\mathbb{R}^n$  is trivial, we conclude that  $\mathbb{R}^n$  is indeed a simply connected space. Furthermore, we can also prove that the 2-sphere, denoted as  $S^2$ , is simply connected because its fundamental group is trivial. In general, the unit sphere  $S^n$  is simply connected if and only if  $n \geq 2$ .

Now, let's delve into this important theorem concerning groups acting on simply connected spaces. For a solid reference, you can consult Armstrong's work on this subject. What this theorem states is as follows: Let  $G$  be a group that acts on a simply connected space  $X$  through homeomorphisms. For every element  $x$  in  $X$ , there exists an open set  $U_x$  such that  $x$  is contained

within this open set, and importantly, the intersection of  $g \cdot U_x$  and  $U_x$  is empty for all non-identity elements  $g$  in the group.

Here, we have a situation where the group  $G$  is acting on the space  $X$ . For a fixed element  $g$ , the action defined by the map is a homeomorphism. Additionally, we have the assumption that for any non-identity element  $g$ , the expression  $g \cdot U_x$  makes sense, and the intersection  $g \cdot U_x \cap U_x$  remains empty.

The crux of this theorem is the claim that the fundamental group of the quotient space  $X/G$ , which is the set of all orbits, is isomorphic to the group  $G$ . This result is established using the theory of covering spaces, and while I won't be going into the proof here, you may have encountered it in your algebraic topology course. We will apply this significant result within our current context.

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Examples:- (i)  $\mathbb{Z}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$   
 $((m_1, \dots, m_n), (x_1, \dots, x_n)) \mapsto (x_1 + m_1, \dots, x_n + m_n)$   
 (ii)  $(\mathbb{R}^n / \mathbb{Z}^n) \cong \mathbb{T}^n$ ,  
 $\mathbb{R}^n / \mathbb{Z}^n$  is homeomorphic to  $\underbrace{S^1 \times \dots \times S^1}_{n\text{-times}}$   
 $n$ -torus

(iii) Hyperbolic Surface  
 let  $g \geq 2$ . The group  $\langle a_1, b_1, a_2, b_2, \dots, a_g, b_g \mid [a_1, b_1] \dots [a_g, b_g] = 1 \rangle$   
 where  $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$ , acts properly discontinuously by isometries on unit disc  $\mathbb{D}^2$  with hyperbolic metric.  
 Thus,  
 $\pi_1 \left( \frac{\mathbb{D}^2}{\langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1] \dots [a_g, b_g] = 1 \rangle} \right)$   
 $= \langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1] \dots [a_g, b_g] = 1 \rangle$   
 & the quotient space  $\frac{\mathbb{D}^2}{\langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1] \dots [a_g, b_g] = 1 \rangle}$  is homeomorphic to closed orientable surface of genus  $g$  where  $g \geq 2$ .

Let's consider the group  $\mathbb{Z}^n$ , which acts on  $\mathbb{R}^n$  through translation. This action is quite straightforward, as it involves shifting points in  $\mathbb{R}^n$  by integer vectors. Notably,  $\mathbb{R}^n$  is a simply connected space, and you can verify that the group  $\mathbb{Z}^n$  satisfies the properties necessary for this action. Therefore, we can conclude that the fundamental group of the quotient space  $\mathbb{R}^n / \mathbb{Z}^n$  is isomorphic to  $\mathbb{Z}^n$ .

What does this quotient space represent? It is homeomorphic to the  $n$ -torus, denoted as  $S^1 \times S^1 \times \dots$ , (up to  $n$  times). Now, let me share another important fact that we will prove later.

Consider the group generated by elements  $a_1b_1, a_2b_2, \dots, a_gb_g$ , with the product of these commutators equaling 1.

But what exactly do we mean by a commutator? The commutator is denoted by the notation  $[a_i, b_i]$ , which is defined as  $a_ib_ia_i^{-1}b_i^{-1}$ . This particular group acts properly discontinuously by isometries on the unit disc equipped with the hyperbolic metric.

In a future class, we will establish that this group indeed acts properly discontinuously by isometries on the hyperbolic plane, or, equivalently, on the unit disc model of the hyperbolic plane. This action is properly discontinuous, and from this property, we can infer that it meets the necessary conditions set forth by our theorem.

Consequently, the fundamental group of the quotient space is isomorphic to this group, which corresponds to a closed orientable surface of genus  $g$ , where  $g \geq 2$ . Since this group acts properly discontinuously on the hyperbolic plane, we can classify it as a Fuchsian group. This classification aligns with the fundamental group of a closed orientable surface of genus  $g$ , where  $g$  is indeed greater than or equal to 2. I will be proving that this group acts properly discontinuously by isometries on the unit disc with a hyperbolic metric in an upcoming class, which will require a detailed proof.