An Introduction to Hyperbolic Geometry

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Module - 1

Lecture - 2

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Hello everyone, and welcome to this insightful lecture on Hyperbolic Geometry. Today, we are going to dive into some fundamental concepts of hyperbolic geometry, an intriguing branch of non-Euclidean geometry. What sets hyperbolic geometry apart is its defiance of the fifth postulate in Euclidean geometry, which leads to a whole new world of geometric relationships and properties.

In this lecture, our focus will be on understanding the hyperbolic plane, and specifically, we'll explore the upper half-plane model of hyperbolic geometry. It's important to note that this is just one of several fascinating models used to represent hyperbolic space, and we will delve into the others in subsequent discussions.

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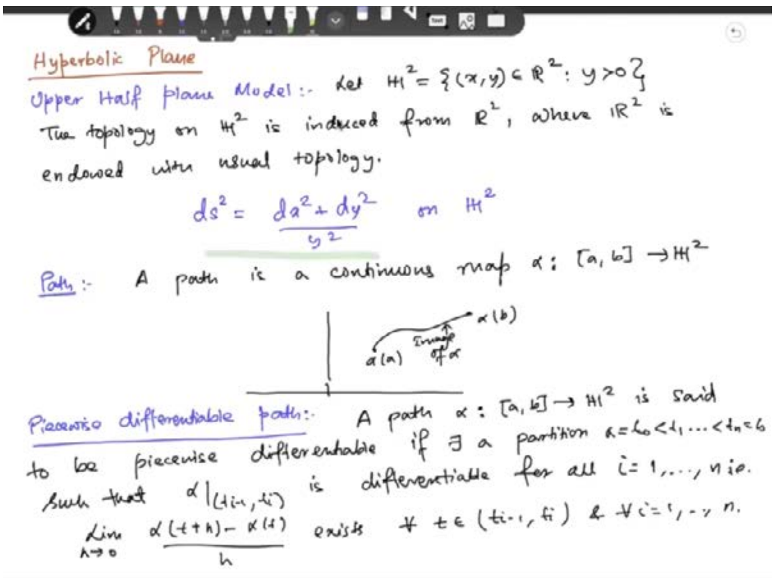

Let's begin by describing the hyperbolic plane, which forms the foundation of hyperbolic geometry, where the fifth axiom of Euclidean geometry no longer holds. Our primary focus in this lecture is to delve into the upper half-plane model of the hyperbolic plane.

So, what exactly is this upper half-plane? It consists of ordered pairs $(x, y) \in R^{2}$, where both x and y are real numbers, and y is always positive. This is what we refer to as the upper half-plane, and to give it structure, we apply a topology on this set. Specifically, we induce the subspace topology from R^{2}. From here on, we will denote the upper half-plane as H^{2}. The topology on H^{2} is, therefore, the subspace topology inherited from R^{2}, where R^{2} itself is equipped with the usual topology.

Now, let’s move forward and define a metric on this upper half-plane, which we will refer to as the hyperbolic metric, or alternatively, the Poincaré metric. To introduce this metric, we’ll first consider the following formula:

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

This equation defines a metric on H^{2}, and we will explore its implications in a moment. But before we proceed with the metric, let's take a step back to clarify the concepts of paths and piecewise differentiable paths on the upper half-plane.

So, what exactly do we mean by a path? A path is simply a continuous map α from a closed interval [a, b] into H^{2}. That is, if we consider the upper half-plane H^{2} and choose two points α(a) and α(b) within this plane, then the image of α, which traces the points between α(a) and α(b), forms the path.

Now, what does it mean for a path to be piecewise differentiable? A path α is said to be piecewise differentiable if there exists a partition of the closed interval [a, b] into sub-intervals such that α, when restricted to each sub-interval, is differentiable. More formally, for each sub-interval $[t_{i-1}, t_i]$, the limit

$$\lim_{h \to 0} \frac{\alpha(t+h) - \alpha(t)}{h}$$

exists for all $t \in [t_{i-1}, t_i]$, and this holds for each i ranging from 1 to n.

This is what we mean when we say a path is piecewise differentiable. Now, with this concept in hand, and using the formula for ds^2, we can compute the length of any given path on the upper half-plane. Let’s move forward to see how that works.

Let’s go through this carefully once again, as it’s essential to understand the formal definition of the hyperbolic length in the context of the upper half-plane model.

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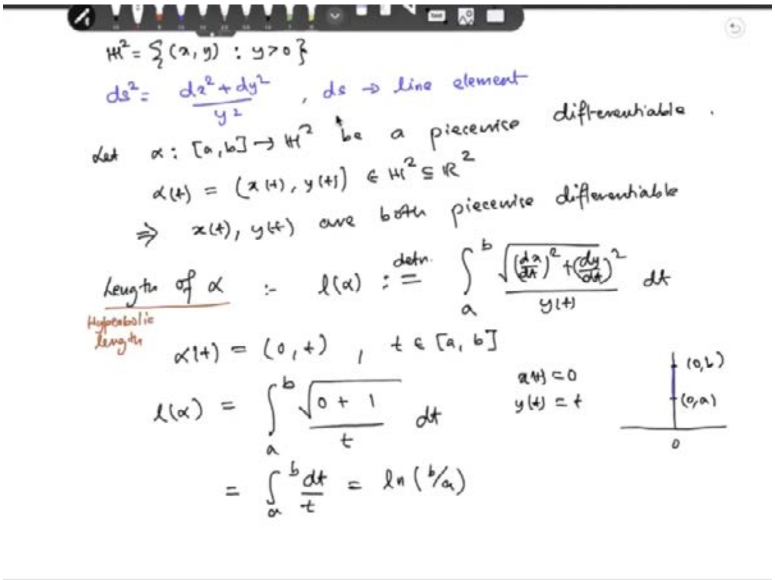

We are given the upper half-plane H^2 and the formula for the line element:

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

Here, ds represents what is called the "line element." Now, suppose we have a piecewise differentiable path α(t) in H^2. This path can be expressed as $\alpha(t) = \big(x(t), y(t)\big)$, where both x(t) and y(t) are functions of t, and α(t) lies in H^2, which, as we know, is a subset of R^2.

Because $H^2 \subseteq R^2$, the image of the path α(t) is given as the ordered pair (x(t), y(t)). Since we are assuming α to be piecewise differentiable, this means that both components x(t) and y(t) are also piecewise differentiable.

Given this path, we can now compute its length, which we denote as L(α). The formula for the length of α is defined as follows:

$$L(\alpha) = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \frac{1}{y(t)} \, dt$$

This integral provides the length of the path α. Since x(t) and y(t) are piecewise differentiable, the terms $\frac{dx}{dt}$ and $\frac{dy}{dt}$ make sense and exist where differentiable, making the integral welldefined. Therefore, this gives us a formal definition of the length of the path α in the hyperbolic geometry context.

To clarify this further, let's consider a specific example: take the path α(t) = (0, t), where t ranges from a to b. Now, in this case, x(t) = 0, so:

$$\frac{dx}{dt} = 0$$

and y(t) = t, so:

$$\frac{dy}{dt} = 1$$

Substituting these values into our formula for the length, we get:

$$L(\alpha) = \int_a^b \frac{1}{t}\, dt$$

which simplifies to:

$$L(\alpha) = \ln\left(\frac{b}{a}\right)$$

Thus, for this particular path, the length of α, from (0, a) to (0, b), is given by $\ln(b/a)$. This length is referred to as the "hyperbolic length" of the path α.

Now, if we were to modify the original formula by removing the y^2 term in the denominator, such that the line element becomes:

$$ds^2 = dx^2 + dy^2$$

we would be dealing with the Euclidean length of α. This is the key distinction between Euclidean and hyperbolic lengths, and it is the presence of the y^2 in the denominator that defines the "hyperbolic" nature of the length in this geometry. Hence, we refer to the length computed in the earlier example as the hyperbolic length, which is different from the Euclidean length.

Now, let's delve deeper into how the upper half-plane model of hyperbolic geometry can be expressed in terms of complex numbers.

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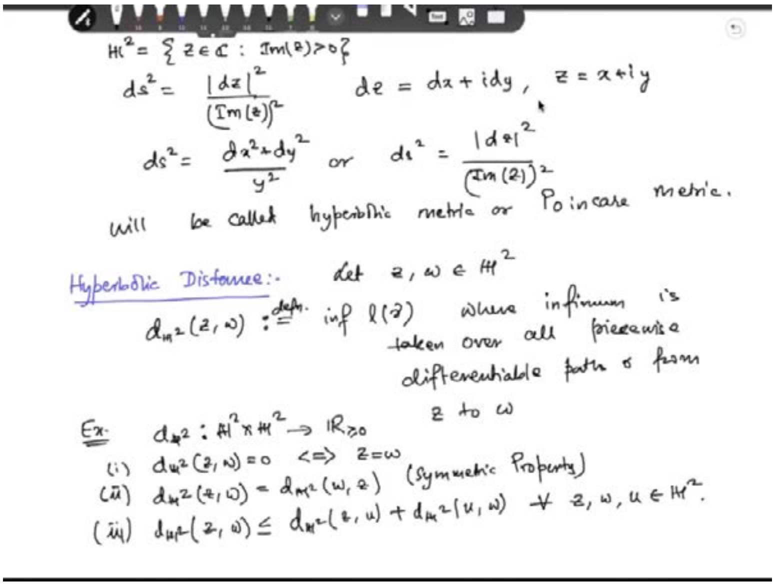

The upper half-plane can be represented as a set of complex numbers. Specifically, it consists of all complex numbers $z \in C$ where the imaginary part of z is greater than zero. So, formally, this set is $\{z \in C \mid \text{Im}(z) > 0\}$. This is a natural extension of our earlier description, where points on the upper half-plane were represented by real coordinates.

In terms of the line element, we can rewrite the formula for the hyperbolic metric using complex notation. The square of the line element ds^2 becomes:

$$ds^2 = \frac{|dz|^2}{\left(\text{Im}(z)\right)^2}$$

Here, dz represents the differential of the complex variable z, and it can be written as $dz = dx + i \ dy$, where z = x + iy. This formula, written in terms of complex numbers, is another way to express the hyperbolic metric, also known as the Poincaré metric.

Next, let's move on to the concept of distance between two points in the upper half-plane using the hyperbolic metric. Suppose we have two points, z and w, both belonging to the upper half-plane $H^{\mathbb{2}}$. The hyperbolic distance between these two points, denoted by $d_{H^{\mathbb{2}}}(z, w)$, is defined as the infimum (i.e., the greatest lower bound) of the lengths of all piecewise differentiable paths γ connecting z and w. Mathematically, this is expressed as:

$$d_{H^{\mathbb{2}}}(z, w) = \inf\{\, L(\gamma) \mid \gamma \text{ is a piecewise differentiable path from } z \text{ to } w\}$$

This definition aligns with the broader concept of distance in metric spaces, but here it's tailored to the hyperbolic context.

To show that d_{H^2} is indeed a metric, we need to verify the three defining properties of a metric:

1. Non-negativity and identity of indiscernibles: The distance between two points z and w in the upper half-plane is non-negative, and $d_{H^2}(z,w) = 0$ if and only if z = w. This ensures that distinct points always have a positive distance between them.

2. Symmetry: The distance function satisfies symmetry, meaning the distance from z to w is the same as the distance from w to z, or formally:

$$d_{H^2}(z,w) = d_{H^2}(w,z)$$

3. Triangle inequality: The hyperbolic distance satisfies the triangle inequality, meaning that for any three points $z, w, u \in H^2$:

$$d_{H^2}(z,w) \leq d_{H^2}(z,u) + d_{H^2}(u,w)$$

This ensures that the direct distance between two points is always less than or equal to the sum of distances through any intermediate point.

These properties together prove that d_{H^2} is a valid metric, turning the upper half-plane H^2 into a metric space.

Finally, it is important to note that the hyperbolic geometry described here violates the fifth postulate of Euclidean geometry (the parallel postulate). This is one of the key distinctions between hyperbolic and Euclidean geometries, and it leads to fascinating geometric properties that are not present in Euclidean spaces.

Let's consider the upper half-plane model of hyperbolic geometry. Suppose we take two points, z and w, that lie on a vertical line in this upper half-plane. Now, what I will demonstrate is the following: if we join z and w by a vertical line segment (let's call this path α) and compare it to any other path γ, where γ is also assumed to be piecewise differentiable, we will find that the length of α is always less than or equal to the length of γ. In other words, α, the vertical line segment, is the shortest path between z and w. This makes α the geodesic between these two points.

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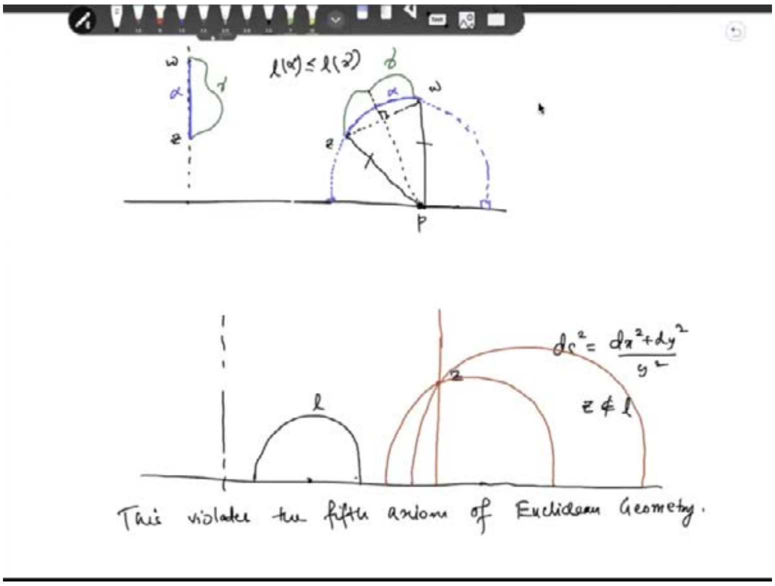

Next, consider the case where z and w do not lie on a vertical line. In this scenario, we can join z and w by a Euclidean line segment and then draw the perpendicular bisector of this line. This bisector will meet the real axis (the x-axis) at some point, which we will call p. The distances from p to z and from p to w will be equal in terms of Euclidean length. Using p as the center, we can now draw a semicircle that passes through both z and w, with a radius equal to the Euclidean distance from p to z.

What I will demonstrate is that this semicircle, which passes through z and w and has its center on the x-axis, represents the shortest path between the two points, just like the vertical line segment in the previous case. If we call this semicircular path α, then for any other path γ, the length of α will be less than or equal to the length of γ. Therefore, the shortest paths in this model of hyperbolic geometry are either vertical lines or portions of semicircles whose centers lie on the real axis.

In Euclidean geometry, straight lines represent the shortest path between two points, and we call these geodesics. Similarly, in hyperbolic geometry, the geodesics are either vertical lines or arcs of semicircles centered on the real axis. These geodesics are the paths of shortest distance between two points in the upper half-plane.

Now, let's explore a fascinating property of hyperbolic geometry: if we take any point z in the upper half-plane and consider any geodesic that does not pass through z, it turns out there are infinitely many other geodesics that pass through z but do not intersect the original geodesic (let's call it l). This is easy to visualize. For example, if l is a vertical line, you can draw infinitely many semicircles centered on the real axis that pass through z without intersecting l.

This property directly contradicts the fifth axiom of Euclidean geometry, also known as the parallel postulate, which states that through a point not on a given line, there is exactly one line parallel to the given line. In hyperbolic geometry, however, there are infinitely many such lines, or in this case, geodesics, that do not intersect the given geodesic l.

In the next lecture, we will learn the concept of isometries in the upper half-plane model and provide a formal proof that the geodesics in this model are indeed either vertical lines or semicircular arcs.