An Introduction to Hyperbolic Geometry
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Lecture - 19
Non-Elementary Fuchsian Groups

Hello, everyone! In our last lecture, we explored the fascinating concept of elementary Fuchsian groups. To recap, elementary Fuchsian groups are defined as those Fuchsian groups that have a finite orbit in the upper half-plane, including its boundary at infinity.

Today, we will shift our focus to identifying non-elementary Fuchsian groups. What do we mean by non-elementary? Essentially, we are looking for groups where every orbit is infinite. A Fuchsian group operates on the hyperbolic plane, and our aim is to discover those Fuchsian groups that ensure each orbit is indeed infinite.

As an illustration, consider PSL(2, Z), which is a discrete subgroup of PSL(2, R). This serves as a prime example of a non-elementary Fuchsian group.

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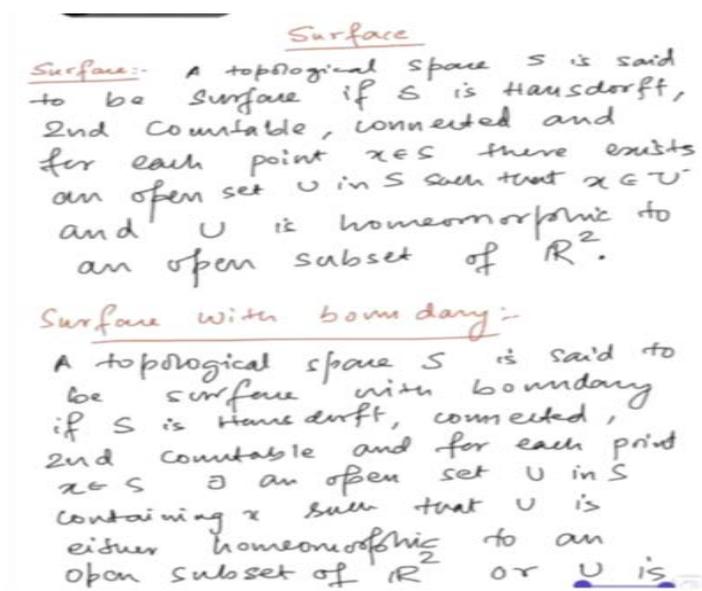
Moving forward, we will demonstrate that if we take a closed orientable surface with a genus of 2 or higher, its fundamental group will constitute a non-elementary Fuchsian group. To grasp the fundamental group of surfaces, we first need to define what we mean by a surface. Therefore,

in this lecture, we will provide definitions and delve into the classification of surfaces. Let's begin!

Let us begin by exploring the definition of a surface. We start with a topological space S. This space is considered a surface if it satisfies the following criteria: it must be a Hausdorff space, meaning that for any two distinct points, there exist disjoint neighborhoods around each point. Additionally, the space must be second countable, which indicates that it has a countable basis. It should also be connected, meaning there are no separate pieces or disjoint subsets within the space.

For each point x in the surface S, there must exist an open set U such that the point x lies within this open set. Furthermore, this open set U should be homeomorphic to an open subset of R^2. If a space is locally homeomorphic to R^2, we refer to it as a surface. This is our definition of a surface.

Now, let's discuss what we mean by a surface with boundary. A topological space is classified as a surface with boundary if it meets similar criteria: it must again be Hausdorff, connected, and second countable. For each point x in this space S, one of the following conditions must hold: either there exists an open set that is homeomorphic to an open subset of R^2, or there exists an open set U that has points in S but also includes points that are not in S, indicating the presence of a boundary.

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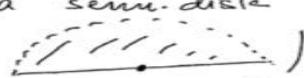
Let's clarify the concept of a semi-disc in R^2. When we refer to a semi-disc, we mean a portion of the x-axis represented as an open interval, combined with a circular disc. Specifically, in a semi-disc, the upper boundary is absent, while the lower boundary is included. This is how we define a semi-disc in R^2. Consequently, if an open subset of S is homeomorphic to either an open subset of R^2 or to a semi-disc in R^2, we categorize the space as a surface with a boundary.

Now, let's move to an example: the unit sphere. We aim to demonstrate that the unit sphere in R^3 qualifies as a surface. To establish this, we need to find an open set for each point that is homeomorphic to an open subset of R^2. The standard approach to prove that the sphere is a surface is through stereographic projection.

Let us define the open set U as S^2 excluding the north pole, which we designate as N = (0, 0, 1). We then apply the stereographic projection.

So, what exactly is stereographic projection? In this method, we consider a point x in U and draw a straight line originating from the north pole N that passes through the point $\bar{x}$. This straight line will intersect the plane z = 0 at some point, meeting the x-y plane. We denote this intersection point as $r(\bar{x})$. Here, r represents the mapping that takes the point $\bar{x}$ to the point r($\bar{x}$). This mapping is referred to as the stereographic projection, and one can demonstrate that r is indeed a homeomorphism from U to R^2.

Similarly, let's define another open set V as the sphere excluding the south pole, which is located at (0, 0, -1). Again, we can apply the stereographic projection to establish a map from V to R^2.

It is important to note that the union of U and V covers the entire sphere S^2, meaning $S^2 = U \cup V$. Therefore, for any point p belonging to S^2, it must lie in either U or V, both of which are homeomorphic to R^2. Hence, we conclude that the sphere is indeed a surface.

Let's now consider an example of a surface with a boundary, the cylinder. The cylinder can be represented as $S^1 \times [0,1]$, where S^1 is the circle and [0, 1] is a closed interval. Now, if you take a point x anywhere on the cylinder, you can find an open neighborhood U around this point, and this neighborhood U will be homeomorphic to an open subset of R^2.

If we take any point on the boundary, say on $S^1 \times \{0\}$ or $S^1 \times \{1\}$, which correspond to the two circular ends of the cylinder, there will exist an open neighborhood of that point which is homeomorphic to a semi-disc. Hence, the cylinder is an example of a surface with two boundary

components.

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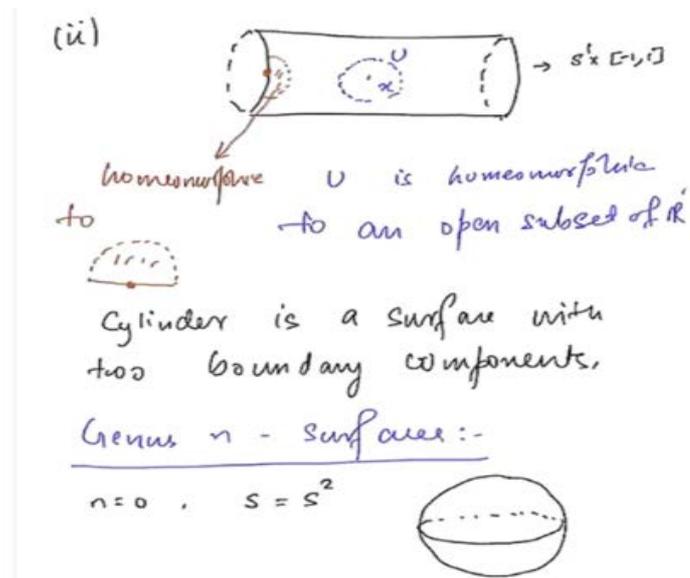
Now, let's define the concept of the genus of a surface. What do we mean by genus? The genus of a surface essentially measures the number of "holes" in it. We'll define this inductively.

For n = 0, meaning the genus is 0, the surface is simply the sphere, or the unit sphere S^2. This surface has no "holes" and is referred to as a surface with genus 0.

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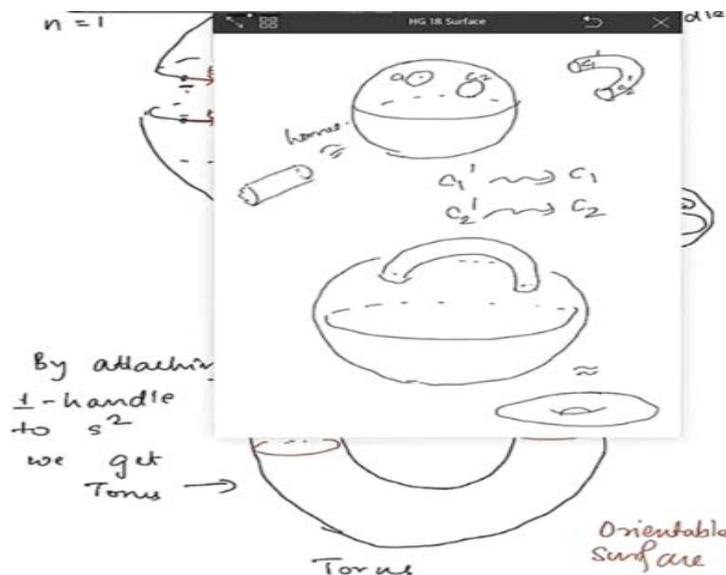
Now, let's move to the case where n = 1, meaning the genus of the surface is 1. Here's what we

do: we take a cylinder and attach it to a sphere in a specific manner. To visualize this, imagine a sphere. Now, create two holes in the sphere by removing two disjoint open discs. This creates two circular boundaries on the sphere, let's call them c_1 and c_2.

Next, take a cylinder. When you remove the open discs from the sphere, you are left with circles c_1 and c_2. The cylinder will also have two boundary components, which we'll call c_1' and c_2'. Now, attach one boundary component of the cylinder c_1' to c_1, and attach the other component c_2' to c_2. Both c_1 and c_1', as well as c_2 and c_2', are homeomorphic to circles, so you can glue them together via a homeomorphism.

Assuming the homeomorphism is orientation-preserving, what you end up with is essentially a surface with a handle attached to the sphere. This surface is homeomorphic to a torus, which is a surface of genus 1.

Now, let's clarify what we mean by a "handle." A handle is simply a homeomorphic copy of a cylinder. So, when we talk about attaching a handle to the sphere, we are referring to removing two open discs from the sphere and attaching a cylinder in their place. Topologically, this process creates something that is homeomorphic to a torus.

In summary, by adding a handle, we transform the sphere into a surface of genus 1, which is topologically equivalent to a torus.

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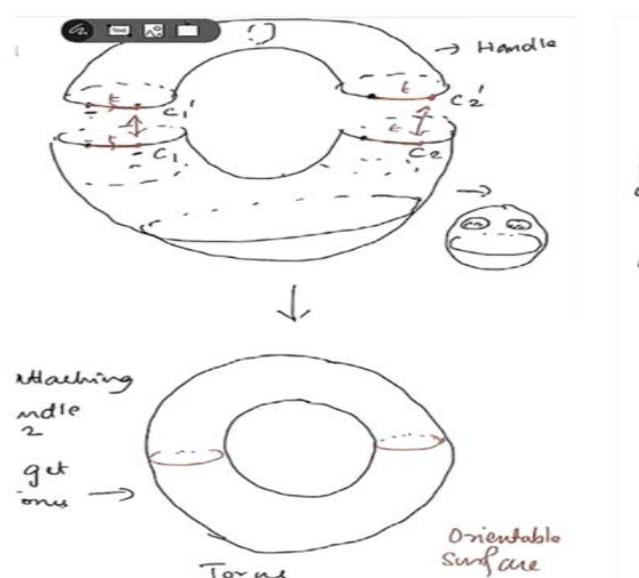
Attaching a handle to a sphere essentially means you're connecting two handles at their

boundary components. In this case, we have two boundary components, let's refer to them as c_1 and c_1'. Imagine the lower part of this structure as a sphere with two holes, and now, we're about to attach the upper part.

There are two ways to perform this attachment, and one of them proceeds as follows: take the boundary component c_1, which belongs to the lower handle, and c_1', the corresponding boundary of the upper handle. Now, attach c_1 to c_1' via the identity map, meaning you're matching them directly and seamlessly.

Similarly, for the second set of boundary components, c_2 and c_2', which also belong to the lower and upper handles, respectively, you repeat the process. Both c_2 and c_2' are homeomorphic to circles, so you can think of them as circles and attach them using the identity map as well.

Once this is done, the result is a smooth structure that forms a torus. So, by attaching just one handle to a sphere, we end up with a torus. This process transforms the simple geometry of a sphere into the more complex topology of a surface with genus 1, which is the torus.

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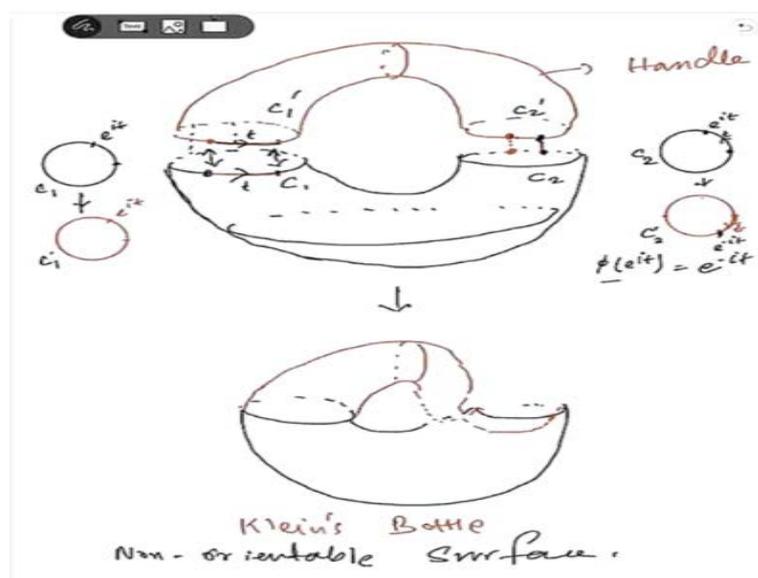
Now, let's consider a variation where one of the attaching maps is orientation-reversing. Once again, we are attaching a handle to a sphere. The lower part is a sphere with two holes, while the upper part is homeomorphic to a cylinder, which forms the handle. In this case, you begin by attaching c_1 to c_1' (the boundary components of the lower and upper parts) using the identity map. So, you match c_1 to c_1' directly using the identity map.

The key difference here occurs when attaching the second boundary components, c_2 and c_2'. Instead of using the identity map, we now use a map of the form $e^{it} \mapsto e^{-it}$. This means that while c_1 and c_1' are still matched directly, c_2 and c_2' are connected through an orientation-reversing map, $\varphi(e^{it}) = e^{-it}$, which essentially "flips" the circle.

Due to this orientation-reversing attachment, the result is no longer a torus. The point c_2', when mapped by φ, is identified with the opposite side of c_2. This creates a very different topological structure, known as the Klein bottle, a famous example of a non-orientable surface. So, by altering the attaching map in this way, we end up with a Klein bottle instead of a torus, highlighting the profound effect of orientation on surface construction.

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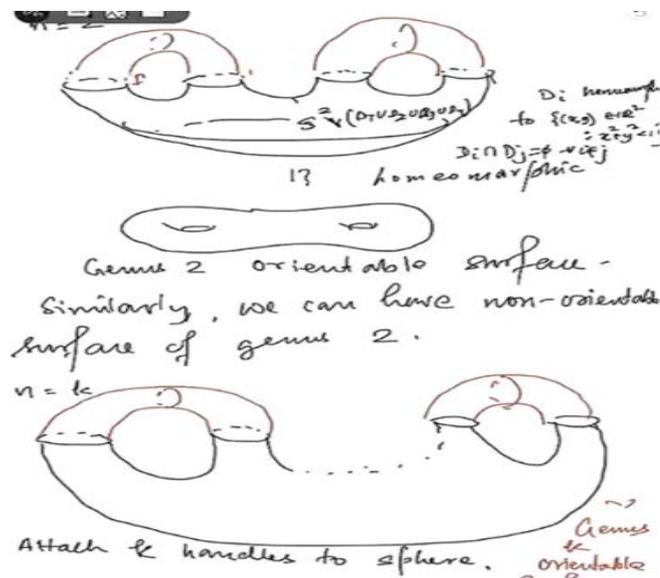
For n = 2, the process continues in a similar fashion. So, what do we do for n = 2? We start with a sphere, and this time we create four holes on the surface, denoted as $D_1 \cup D_2 \cup D_3 \cup D_4$. Here, each D_i is homeomorphic to an open disc, and they are disjoint from one another, meaning that $D_i \cap D_j = \emptyset$, for all $i \neq j$. Essentially, we are making four holes in the sphere.

Now, we attach two handles to this sphere, one to connect the boundary components of two of the holes, and another handle for the other two. If the attaching maps are orientation-preserving homeomorphisms, the resulting surface will be orientable. So, when the maps preserve orientation, we get a surface that can be visualized as a sphere with two handles, or what we call a genus-2 orientable surface.

The term "genus 2" indicates that we have attached two handles to the sphere. Similarly, you can have a non-orientable surface of genus 2, though we won't draw it here. For general n = k, you continue the process, attaching k handles to the sphere to form a surface of genus k.

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Similarly, we can have genus n non-orientable surface.

By genus of a surface, we mean number of handles attached to the sphere to get that surface.

Quotient Topology:-

Let $(x, \mathcal{I})$ be a top. space & Y be a set. Let $f : X \to Y$ be a surjective map.

$$\mathcal{I}_Y = \{ v \subseteq Y : f^{-1}(v) \in \mathcal{I} \}$$

$\mathcal{I}_Y$ forms a topology in Y & is called quotient topology

(*) $f : (x, \mathcal{I}) \to (Y, \mathcal{I}_Y)$ is a continuous map.

Similarly, we can define surfaces of genus n, or non-orientable surfaces. Now, let's formalize this concept a bit more rigorously by considering how to obtain a surface as a quotient space. To do this, we need to first understand the definition of quotient topology.

So, what do we mean by quotient topology? Let X be a topological space and Y be a set. Now, let's consider a surjective map f from X to Y, and on Y, we wish to define a topology in such a way that f remains continuous. The most natural approach is as follows: we take the collection of subsets of Y such that the inverse image of these subsets under f are open in X.

More formally, for any subset $V \subset Y$, we define V to be open in Y if $f^{-1}(V)$ is open in X, where $f^{-1}(V) \in \tau$, the topology on X. This collection of subsets forms a topology on Y, and this topology is called the quotient topology. Under this topology, the map f is guaranteed to be continuous.

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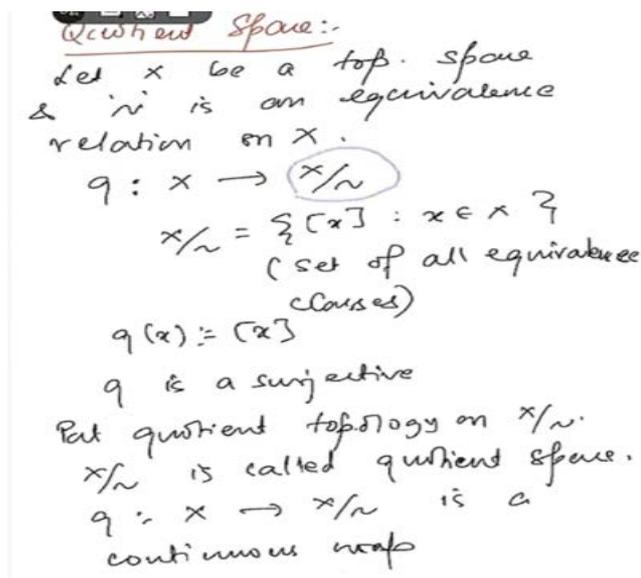
Now, let's consider a topological space X equipped with an equivalence relation. Suppose we define an equivalence relation on X, which naturally leads to the construction of a quotient space. We then have a quotient map $q: X \rightarrow X/\sim$, where $X/\sim$ is the set of equivalence classes of X under the given relation.

On this quotient space, we can impose the quotient topology, ensuring that the map q remains continuous. This means that q will be a continuous map from X to the quotient space $X/\sim$. Thus, in the quotient space, the topology defined is precisely the quotient topology that makes q continuous.

For example, let's consider a square and identify its opposite sides. So, imagine a square where the bottom side is identified with the top side. Similarly, you can define an equivalence relation on this square by identifying points on opposite sides: (s, 0) is identified with (s, 1) for the horizontal edges, and for the vertical edges, (0, t) is identified with (1, t). This defines an equivalence relation on the space X, and the quotient space you obtain by identifying these opposite edges can be shown to be homeomorphic to a torus.

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This is a concept you may have encountered before in a topology course, starting with a unit square and identifying opposite sides results in a torus. Geometrically, this torus can also be described as a surface of revolution. The equation of the torus can be derived by defining a map f from this unit square to R^3.

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Now, how do we define this map f? Consider R^3 and take a circle of radius r whose center lies on the x-axis. The distance of this center from the origin is R. Then, by rotating the circle around the z-axis, you obtain the familiar shape of the torus, often called the "doughnut shape." This is

the geometry behind constructing a torus from a square.

We refer to T^2 as the round torus. Here, X is the unit square, and we have a map from X to the torus. If you want to demonstrate that the quotient space is homeomorphic to the torus, this is the approach to follow. You start with a map from X to the torus and a quotient map q from X to the quotient space. From this, there arises a natural map $\overline{f}$ from the quotient space to the torus. One can prove that this map $\overline{f}$ is indeed a homeomorphism.

This result shows that the quotient space and the torus are homeomorphic, as written here. Furthermore, the torus can also be represented as $S^1 \times S^1$, which means the product of two circles. Similarly, you can prove that $S^1 \times S^1$ is homeomorphic to the torus. In fact, you also have a map, g, from X (the unit square) to $S^1 \times S^1$, further reinforcing the connection between these topological spaces.

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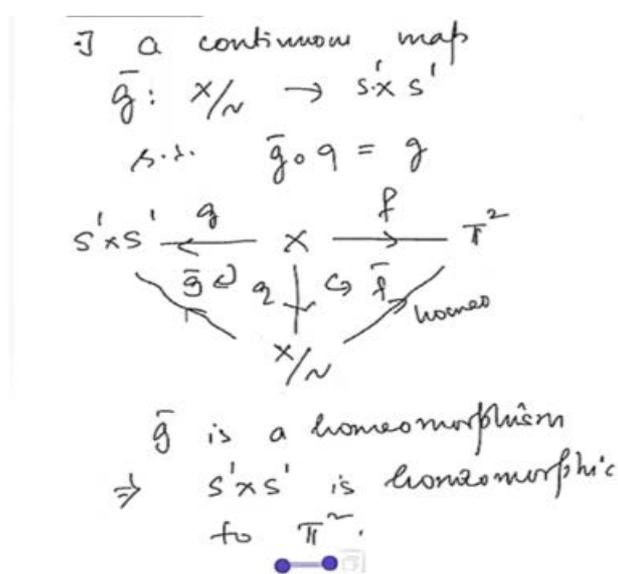
There is also a continuous map $\overline{g}$ from the quotient space to $S^1 \times S^1$, such that when $\overline{g}$ is composed with the quotient map, it equals g. One can prove that this map $\overline{g}$ is indeed a homeomorphism. Therefore, the quotient space is homeomorphic to the torus, and similarly, $S^1 \times S^1$ is homeomorphic to the torus. Since we've already established this, I won't repeat the details here.

Another important concept to discuss is the connected sum. I will elaborate on this shortly. Let's begin by considering two topological spaces, X and Y. Suppose we have A as a subspace of X

and B as a subspace of Y. If there exists a homeomorphism φ from A to B, we can form the disjoint union of X and Y.

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Our goal is to construct a new space from X and Y by attaching A to B. To achieve this, we need to define a relation on the disjoint union. Specifically, we say that a is related to $\varphi(a)$ for every a in A.

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In our discussion, we define the relation such that a is related to φ(a), which establishes our relationship. Next, we form the quotient space based on this relation. This is a straightforward

approach to constructing the space.

To illustrate, let's visualize our spaces: here is space X alongside subspace A, with a map φ connecting A to B. When you take a point from set A and its corresponding point φ(a), this relation holds true. Therefore, we create this quotient space and endow it with the quotient topology.

I will denote this newly formed space as follows. The map φ is referred to as the attaching map. For instance, let's consider X to be a cylinder and Y to be another copy of the cylinder. We can designate A as $c_1 \cup c_2$ and B as $c_1' \cup c_2'$. In this scenario, what does the map φ look like? It arises from the identity map, which takes c_1 to c_1' and c_2 to c_2'. Here, φ is simply the identity, leading us to conclude that the space we construct is, in fact, a torus.

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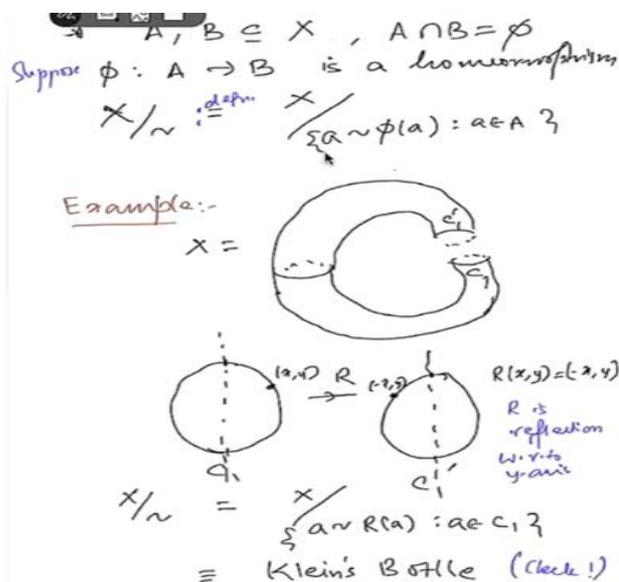
Earlier, we discussed a crucial point. Instead of employing the identity map in this context, if we opt for an orientation-reversing map, the resulting space transforms into the famous Klein bottle. This is a significant concept, and while I won't delve into the details again, it's important to recognize how the nature of the attaching map directly influences the resulting topology of the space.

Now, let's delve into the definition of the connected sum. Consider two surfaces, S_1 and S_2, along with open sets D_i in each surface, such that D_i is homeomorphic to an open disc in R^2. Visualize this: the closure of D_i, denoted as $\overline{D_i}$, represents a closed disc.
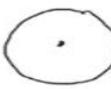
The next step is to remove this open disc from each surface S_i, leading to $S_i - D_i$ for both $i = 1$ and $i = 2$. It's essential to note that the boundary of D_i, denoted as ∂D_i, is the topological boundary of this open disc. In this context, when we refer to "boundary," we mean the topological boundary.
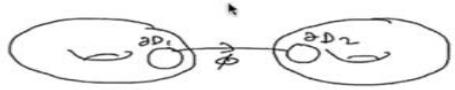
Thus, ∂D_i corresponds to the topological boundary of D_i for both $i = 1$ and 2. This boundary is homeomorphic to a circle. For instance, if we take S_1 to be a torus and S_2 to also be a torus, removing the open disc from each torus will yield a torus with one boundary component, and

we will have another copy of this structure.

So, what do we do next? We attach the boundary ∂D_1 to the boundary ∂D_2 using a homeomorphism. This process is precisely what we refer to as the connected sum. The map ϕ serves as the homeomorphism from ∂D_1 to ∂D_2, allowing us to connect these two surfaces. In this case, we are essentially attaching a torus with one boundary component to another torus that also has a boundary component.

By making this connection, we create a new surface known as the connected sum, which is actually a genus 2 surface. Importantly, this process can be repeated for any surface of genus n. Each time, by attaching boundary components via homeomorphisms, we can construct more complex surfaces, expanding our understanding of topological structures.

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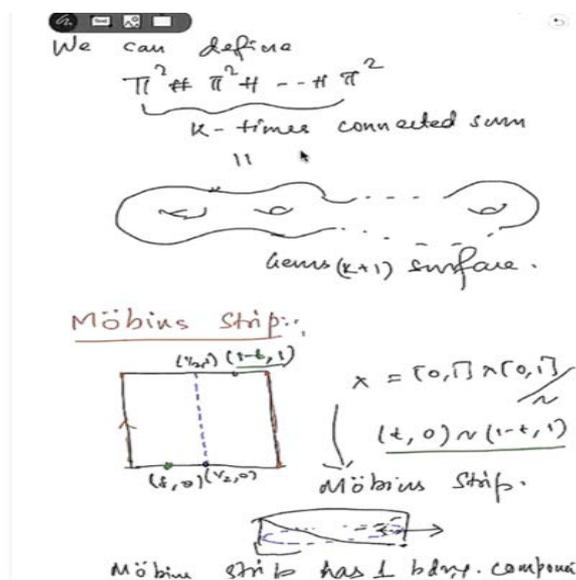
You can repeat this process multiple times, effectively performing the connected sum operation on a torus. So, if we take a torus and connect it with another torus, and continue this process k times, we will obtain a genus k + 1 surface.

Now, let's define what a Möbius strip is. We begin with a unit square, and the identification process involves connecting the point (t, 0) to the point (1 - t, 1). This recipe for identification is crucial, as it determines the structure of the resulting space.

When you apply this identification, the quotient space you create is a Möbius strip. It can be shown that this Möbius strip possesses exactly one boundary component. In contrast, a cylinder

has two boundary components, while the Möbius strip uniquely has only one. This distinction highlights the fascinating properties of these different surfaces in topology.

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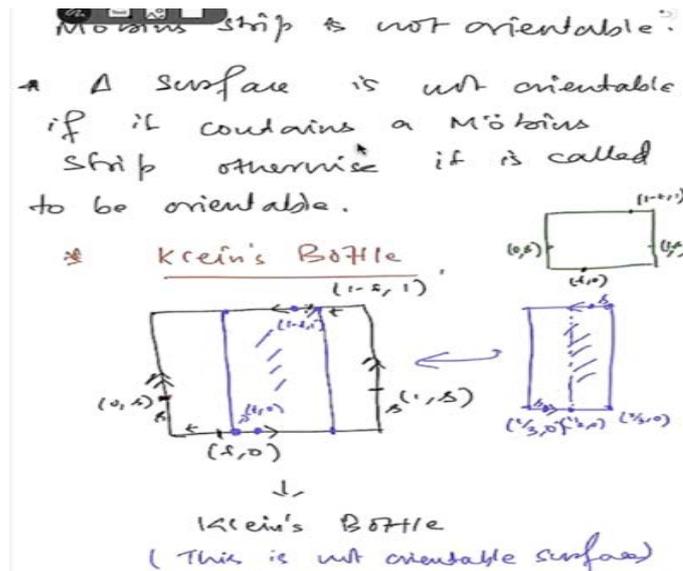
If you're familiar with the concept of orientability, it's essential to note that the Möbius strip is a non-orientable surface. To further elaborate, we can define non-orientability as follows: a surface is deemed non-orientable if it contains a Möbius strip; otherwise, it is referred to as orientable.

Now, let's examine the Klein bottle. As I've demonstrated, if we take a Klein bottle and consider its design, it can be represented by a square. In this representation, the point (0, s) is identified with (1, s), while the point (t, 0) is identified with (1 - t, 1). It's crucial to highlight that the identification occurs solely along the boundary of the square. When you take the quotient space resulting from this identification, you will obtain the Klein bottle.

Now, let's consider a rectangle where one point is at (1/3, 0) and the interval extends to (2/3, 0). This rectangle, when manipulated appropriately, represents a Möbius strip. Since the Möbius strip is included within the Klein bottle, we conclude that the Klein bottle itself is a non-orientable surface. This connection illustrates the intricate relationships between different surfaces in topology.

Let's delve into the classification theorem for closed surfaces. First, we need to define what we mean by a closed surface. A surface S is considered closed if it is both compact and devoid of

any boundaries; here, "boundary" refers specifically to the topological boundary. In simpler terms, a closed surface is compact and has no edges or boundaries.

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Now, turning our attention to the classification of orientable closed surfaces, we can make a significant observation: any closed orientable surface is homeomorphic to one of the following types. It could either be a sphere, which corresponds to a genus 0 surface, or a torus, representing a genus 1 surface. Furthermore, we can have a genus 2 surface, which is essentially a connected sum of two tori.

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In general, we can classify these surfaces as a genus g surface, where g represents the number of "holes" or handles in the surface. This classification offers a structured way to understand and categorize orientable closed surfaces in topology.

Now, let's explore the classification of non-orientable closed surfaces. To understand this classification, we first need to clarify the concept of the real projective plane. Imagine taking a unit disc, which has a boundary, and consider the identification occurring only along this boundary. Specifically, the point (x, y) on the boundary is identified with the point (-x, -y). For any point satisfying the equation $x^2 + y^2 = 1$, the identification is exclusively along the boundary, meaning that both (x, y) and (-x, -y) lie on the unit circle S^1.

This leads us to define an equivalence relation on this disc, where we state that the point (x, y) is related to the point (-x, -y). The resulting quotient space from this identification is denoted as RP^2, which stands for the real projective plane.

It can be rigorously proven that RP^2 is indeed a surface, and furthermore, it is classified as a non-orientable surface, an interesting exercise to undertake. Moreover, the Klein bottle can be understood as the connected sum of two real projective planes, symbolically expressed as $RP^2 \# RP^2$. This relationship highlights the intricate structure and nature of non-orientable closed surfaces within topology.

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Now, let's delve into the classification theorem for non-orientable closed surfaces. According

to this theorem, any closed non-orientable surface is homeomorphic to one of the following types: it can either be the real projective plane, denoted as RP^2, or it can be the connected sum of two real projective planes, which results in the Klein bottle. Furthermore, it may also be represented as the k-times connected sum of the real projective plane, RP^2, where k indicates the number of connected sums.

In summary, this classification theorem encapsulates the different forms that non-orientable closed surfaces can take. Additionally, in an upcoming lecture, we will explore the fascinating concept of the fundamental group of surfaces, which further enriches our understanding of topological spaces.