**An Introduction to Hyperbolic Geometry**

**Prof. Abhijit Pal**

**Department of Mathematics and Statistics**

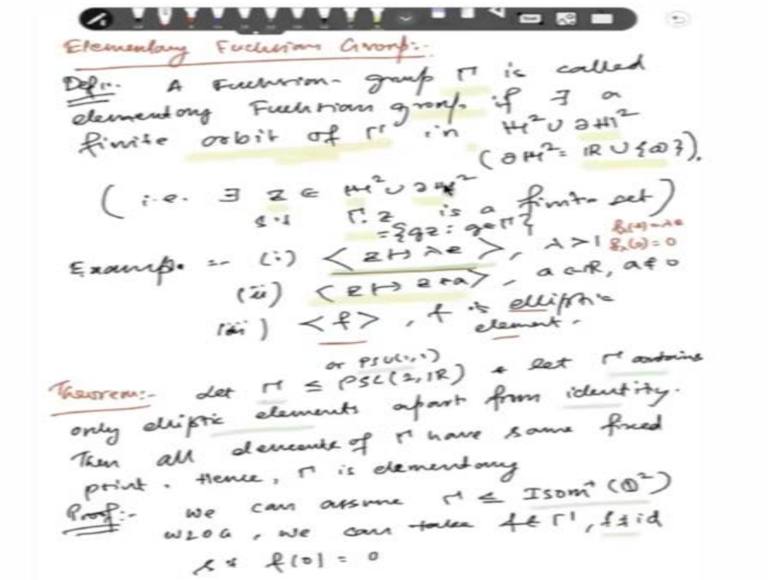
**Indian Institute of Technology – Kanpur**

**Lecture – 18**

**Elementary Fuchsian Groups**

Hello! In today's lecture, we will be exploring the concept of elementary Fuchsian groups. These are a special class of Fuchsian groups with distinct characteristics. Specifically, a Fuchsian group is classified as elementary if it has a finite orbit in the hyperbolic plane, including its boundary at infinity. So let us begin.

**(Refer Slide Time: 00:43)**




Let's start by defining what an elementary Fuchsian group is. Here, we'll focus on the group $\mathrm{PSL}(2, R)$, but the same definition applies to $\mathrm{PSU}(1,1)$ as well. The group $\mathrm{PSU}(1,1)$ acts on the unit disk model of the hyperbolic plane through isometries, and it preserves the orientation of the unit disk. While I will primarily discuss the elementary Fuchsian group within the context of $\mathrm{PSL}(2, R)$, it's important to remember that the concepts are equally valid for $\mathrm{PSU}(1,1)$, since both groups are isomorphic to each other.

Now, a Fuchsian group $\Gamma$ is called an elementary Fuchsian group if it has a finite orbit in the upper half-plane, $H^2$, together with its boundary at infinity. What do we mean by the boundary of the upper half-plane? It refers to $R \cup \{\infty\}$.

To clarify, if you take any element $t \in \Gamma$, it acts on the upper half-plane via Möbius transformations, which can be extended to act on $R \cup \{\infty\}$.

Let me repeat the definition of an elementary Fuchsian group: A Fuchsian group is considered elementary if there exists a point z, either in the upper half-plane or on its boundary $R \cup \{\infty\}$, such that the orbit of z under the action of Γ, denoted $\Gamma \cdot z$, is a finite set. To put it more formally, the orbit is the set $\{g \cdot z \mid g \in \Gamma\}$.

Now, let's look at some examples to better understand this concept.

Consider a cyclic subgroup of $\mathrm{PSL}(2, R)$ generated by a hyperbolic isometry of the form $z \mapsto \lambda z$, where λ > 1. Why is this an elementary Fuchsian group? Well, if we take this hyperbolic isometry, say $f_\lambda(z) = \lambda z$, the fixed points of $f_\lambda$ are 0 and ∞. Applying $f_\lambda$ to 0 gives us 0, and any other element of this group will have the form $z \mapsto \lambda^n z$, which also fixes 0.

Therefore, this cyclic subgroup of $\mathrm{PSL}(2, R)$, generated by the transformation $z \mapsto \lambda z$, is an elementary Fuchsian group because the orbit of 0 is a finite set.

Let's look at another example. Consider the parabolic isometry $z \mapsto z + a$, which fixes the point ∞. Any element of this group will be of the form $z \mapsto z + na$, where $n \in Z$, and this transformation will also fix ∞. So, the orbit of ∞ is just the singleton set {∞}, making this another example of an elementary Fuchsian group.

Thus, we again find that this forms an elementary Fuchsian group. Now, if we consider f as an elliptic element, it will fix a point in the upper half-plane, meaning that every element in this group will also fix that same point. Since we are dealing with a cyclic group, it follows that all elements of this group will fix this point, resulting in the orbit of that point being a singleton set. Consequently, we can affirm that this group is indeed an elementary Fuchsian group.

Now, let's turn our attention to an important theorem.

Let's consider a subgroup of $\mathrm{PSL}(2, R)$ and assume that this subgroup Γ contains only elliptic elements, aside from the identity. This means that all elements of Γ, except for the identity, are elliptic elements. We will now prove that all elements of Γ share the same fixed point. Consequently, since all the elements fix the same point, the subgroup Γ must be an elementary group. The orbit of this fixed point will thus be a singleton set, confirming that Γ is indeed an elementary group.

In this context, we will consider the unit disk model of the hyperbolic plane. We can assume Γ to be a subgroup of $\mathrm{PSU}(1,1)$. It's worth noting that $\mathrm{PSL}(2, R)$ is isomorphic to $\mathrm{PSU}(1,1)$, so we can also represent Γ as a subgroup of $\mathrm{PSU}(1,1)$. The group $\mathrm{PSU}(1,1)$ comprises the orientation-preserving isometries of the unit disk model of the hyperbolic plane, thus confirming that Γ is indeed a subgroup of that group.

Now, let's take an element $f \in \Gamma$, which is not equal to the identity. By our earlier assumption, f is an elliptic element, meaning it will fix some point within the unit disk. Without loss of generality, we can assume that the fixed point of f is 0. Since we are working with this unit disk model, let's denote the fixed point of f as $z_0$, which lies within the unit disk.

To simplify our discussion, we can define an isometry of the unit disk that maps $z_0$ to the origin; let's denote this isometry as g. Now, if we take the transformation $gfg^{-1}$, this isometric operation will also fix the point 0, or the origin. Therefore, for the sake of convenience, we will assume that f fixes 0.

**(Refer Slide Time: 09:55)**


Thus, we find that f fixes 0, which implies that f(z) will be of a specific form since f(0) = 0. Consequently, this leads to the conclusion that b = 0. Therefore, the corresponding matrix for f will be a diagonal matrix, which is why I've expressed it in this form. Since f fixes 0, it can indeed be represented as such. Thus, the matrix corresponding to f can be written as:

$$\begin{pmatrix} u & 0 \\ 0 & \bar{u} \end{pmatrix}$$

Now, let's consider an isometry of this unit disk, where g belongs to Γ. We will assume that g is not the identity. The matrix representation of g takes the form:

$$\begin{pmatrix} a & c \\ \bar{c} & \bar{a} \end{pmatrix}$$

where a and c are complex numbers satisfying $|a|^2 - |c|^2 = 1$. Hence, we can express g(z) in this manner.

Next, we will compute the trace of the commutator, denoted as $[f, g] = fgf^{-1}g^{-1}$. This commutator will also result in a Möbius transformation, and we can find its corresponding matrix to compute the trace. The trace of the commutator turns out to be:

$$\mathrm{trace}([f, g]) = 2 + 4|c|^2 \cdot \mathrm{Im}(u)^2$$

This is a straightforward computation, and we note that we have previously assumed that Γ consists solely of elliptic elements, apart from the identity. Since both f and g belong to Γ, their commutator must also belong to Γ. Thus, this commutator can only be either the identity or an elliptic element, which is why the trace of this matrix cannot exceed 2.

If the trace were greater than 2, it would indicate the presence of a hyperbolic element, which is not allowed under our current assumptions. Thus, the trace must be either equal to 2 (if the commutator is the identity) or less than 2 (if it is an elliptic element). Therefore, we conclude that the trace of this matrix is:

$$\mathrm{trace}([f, g]) \leq 2$$

Returning to our earlier equation, since the left-hand side is less than or equal to 2, we deduce that the term $4|c|^2 \cdot \mathrm{Im}(u)^2$ must be less than or equal to 0.

Since we have a square term involved, this implies that either c = 0 or $\mathrm{Im}(u) = 0$. If $\mathrm{Im}(u) = 0$, it indicates that $u = \bar{u}$, and given that $u\bar{u} = 1$, we conclude that the matrix corresponding to f would simply be the identity matrix. This presents a contradiction, as the identity matrix does not belong to $\mathrm{PSU}(1,1)$.

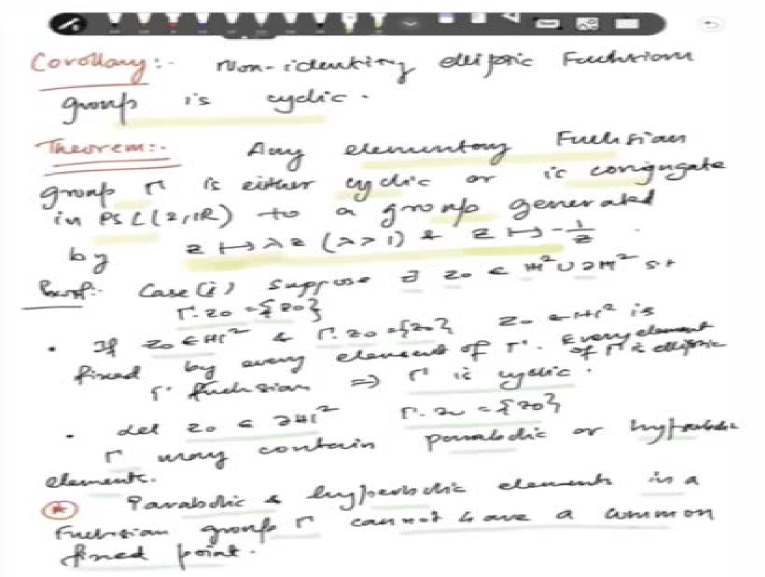
Thus, we have reached a contradiction, as denoted by the symbol for contradiction. This means that the assumption $\mathrm{Im}(u) = 0$ must be incorrect, leading us to conclude that c = 0. Given that c = 0, we can revisit our expression for g:

$$g(z) = \frac{az}{|a|^2}$$

From this, it follows that g(0) = 0. Consequently, we have established that the fixed point of g is merely the singleton set $\{0\}$. Since we have shown that g was an arbitrary element of Γ and that g(0) = 0, we can conclude that all elements of Γ indeed have the same fixed point, confirming that Γ is an elementary group.

Furthermore, if we consider Γ as a Fuchsian group, that is, discrete and consisting solely of elliptic elements, then it follows that Γ must be cyclic.

**(Refer Slide Time: 17:58)**




Thus, we conclude that a non-identity element in a Fuchsian group is cyclic. If we consider a Fuchsian group Γ that contains a non-identity elliptic element, it follows that all elliptic elements will fix a single point. This means that the fixed points of all the isometries are the same, leading to the conclusion that all the elements indeed commute with one another. As we established in our previous class, this implies that the group must be cyclic. This gives us our first corollary.

Now, let's move on to the next theorem, which is particularly important as it classifies all elementary Fuchsian groups. This theorem states that any elementary Fuchsian group Γ is either cyclic or conjugate in PSL$(2, R)$ to a group generated by the hyperbolic isometry $z \mapsto \lambda z$, where λ > 1, and the map $z \mapsto -\frac{1}{z}$. Therefore, if we take the group generated by these two transformations, it will form a Fuchsian group.

In essence, any elementary Fuchsian group that is not cyclic will be conjugate to a group generated by these two elements. The proof of this theorem is conducted on a case-by-case basis.

Case 1: Suppose there exists an element $z_0$ in the upper half-plane union the boundary such that the orbit of $z_0$ is merely the singleton point $z_0$. Here, two subcases arise. First, if $z_0$ belongs to the upper half-plane, then the orbit of $z_0$ under $\Gamma$, denoted $\Gamma \cdot z_0$, is the singleton set $\{z_0\}$. This indicates that every non-identity element of $\Gamma$ fixes the point $z_0$. Consequently, all non-identity elements of $\Gamma$ are elliptic, and thus, by our earlier results, $\Gamma$ must be a cyclic group, and we have completed this case.

Next, if we assume that $z_0$ lies on the boundary and we have also assumed that its orbit is a singleton set, we must consider the possibility that $\Gamma$ may contain parabolic or hyperbolic elements. To address this, we will rule out some cases; first, let's prove an important observation: parabolic and hyperbolic elements in a Fuchsian group cannot share a common fixed point. This observation is crucial for our analysis.

**(Refer Slide Time: 21:46)**


Let us begin by proving this observation. Suppose we have two elements: one is parabolic, and the other is hyperbolic, both fixing the same point on the boundary. Without loss of generality, we can take that point to be infinity. Let g be a hyperbolic element and f be a parabolic element in $\Gamma$, both of which fix infinity. It's important to note that, by our assumption, this orbit is simply the singleton point infinity.

Furthermore, we can express this hyperbolic element, which fixes infinity, in the form $z \mapsto \lambda z$, where λ > 1, and let the parabolic isometry be of the form $z \mapsto z + a$. Therefore, we have $g(z) = \lambda z$ and f(z) = z + a. Now, for each integer n, we consider the conjugate $g^{-n} f g^{n} z$. Upon computing this, we find that it simplifies to $z + \frac{a}{\lambda^n}$, which approaches z as n tends to infinity.

This result demonstrates that the sequence of isometries converges pointwise to the identity. Moreover, since all the elements in this sequence belong to Γ, and given that Γ is discrete, the elements $g^{-n} f g^{n}$ are distinct. Therefore, this sequence cannot eventually become constant, leading to a contradiction. Consequently, we conclude that parabolic and hyperbolic elements in a Fuchsian group cannot share a common fixed point.

From this, we deduce that the elements of Γ are either parabolic or hyperbolic. Now, let us suppose that all elements of Γ are hyperbolic. We can take two elements f and g that belong to Γ, and again we have assumed that this orbit is equal to the singleton set. Without loss of generality, we can assume this point $z_0$ to be infinity.

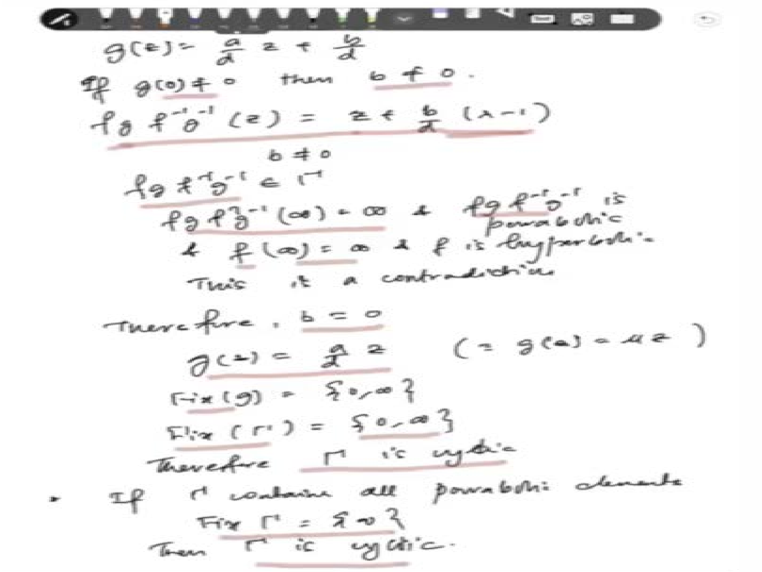
We can define an isometry that maps $z_0$ to infinity. Suppose $f(z_0) = \infty$; then we can construct a group $\Gamma_1$ that is conjugate to Γ using the map f. If all elements of Γ are hyperbolic and $\Gamma \cdot \infty = \infty$, we can once again take two elements f and g from Γ and assume, without loss of generality, that f has the form f(z) = λ z.

We can again perform a conjugation of f that will fix the points at infinity and zero. This mapping will be of the form $z \mapsto \lambda z$. During this conjugation, we are also modifying the group itself, and as per our theorem, it will be conjugate in $\mathrm{PSL}(2, R)$ to a group generated by $z \mapsto \lambda$ z and $z \mapsto -\frac{1}{z}$. Therefore, we can safely assume that $f(z) = \lambda z$.

Now, since g also fixes the point at infinity, if we assume that g(z) is represented by the Möbius transformation $\frac{az+b}{cz+d}$, the condition $g(\infty) =$ implies that c = 0. Thus, we can express $g(z) = \frac{a}{d} z + \frac{b}{d}$, confirming that g is not equal to the identity.

Now, let's consider the scenario where g(0) is not equal to zero. This implies that b is also not equal to zero. If we examine the commutator $[f, g] = f g f^{-1} g^{-1}$ at a point z, we find that it can be expressed as $\frac{z+b}{d}(\lambda - 1)$. Given that b is not equal to zero and since fg belongs to Γ, this commutator must also belong to Γ. Both f and g fix infinity, indicating that the commutator can also fix infinity. However, the structure we have established shows that b being non-zero leads us to conclude that this is indeed a parabolic isometry.

**(Refer Slide Time: 27:43)**



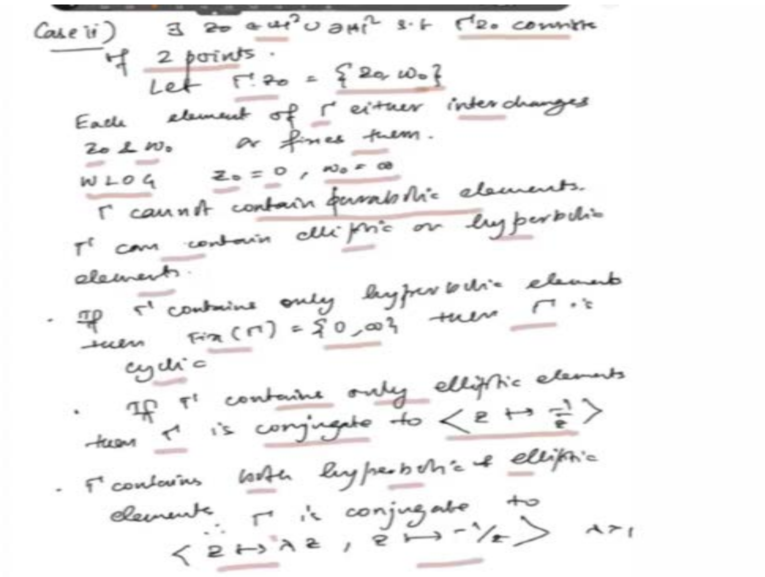

Thus, we find ourselves with a parabolic isometry fixing the point at infinity, alongside a hyperbolic isometry, which also fixes infinity. Since both elements belong to Γ, and given that Γ is a Fuchsian group, it is impossible for both a parabolic and a hyperbolic isometry to fix the same point at infinity. This contradiction proves that b must be equal to zero.

With b = 0, we can conclude that g(z) takes the form $\frac{a}{d}z$. This implies that the fixed points of g are both 0 and infinity. Initially, we started with two elements, where f(z) is of the form λ z, and g can be any arbitrary non-identity element. We have now established that g(z) is of the form $\frac{a}{d}z$, meaning all hyperbolic isometries in this group Γ fix both 0 and infinity.

Consequently, since every element fixes 0 and infinity, they will commute with each other. Referring back to our previous result, this leads us to conclude that Γ must be a cyclic group. Therefore, we have proven that if all elements of Γ are hyperbolic, then Γ is indeed a cyclic group.

Now, let's consider the case where Γ consists entirely of parabolic elements. Once again, without loss of generality, we can assume that $\Gamma \cdot \infty = \infty$, meaning the fixed point of Γ is just infinity. By applying our previous results, we can also demonstrate that this Γ is cyclic.

**(Refer Slide Time: 31:17)**



Case ii) $\exists\, z_0 \in \mathbb{H}^2 \cup \partial\mathbb{H}^2$ s.t. $\Gamma \cdot z_0$ consists of 2 points.

Let $\Gamma \cdot z_0 = \{z_0, w_0\}$

Each element of $\Gamma$ either interchanges $z_0$ & $w_0$ or fixes them.

WLOG $z_0 = 0$, $w_0 = \infty$

$\Gamma$ cannot contain parabolic elements.

$\Gamma$ can contain elliptic or hyperbolic elements.

- If $\Gamma$ contains only hyperbolic elements then $\mathrm{Fix}(\Gamma) = \{0, \infty\}$ then $\Gamma$ is cyclic
- If $\Gamma$ contains only elliptic elements then $\Gamma$ is conjugate to $\langle z \mapsto \frac{-1}{z} \rangle$
- $\Gamma$ contains both hyperbolic & elliptic elements $\therefore$ $\Gamma$ is conjugate to $\langle z \mapsto \lambda z,\ z \mapsto -1/z \rangle$ $\lambda > 1$

Now, let’s proceed to the next case. Suppose we have a point z0 in the space, which includes the boundary of the upper half-plane, such that the orbit consists of only two points. Specifically, let us denote these points as z0 and w0. It’s important to note that z0 will automatically belong to the orbit $\Gamma \cdot z_0$ because Γ contains the identity element.

Since we are assuming that the orbit contains only the two points z0 and w0, each element of Γ either interchanges z0 and w0 or fixes them. We can conveniently take a conjugate of the group Γ such that this orbit can be represented as 0 and infinity. Thus, without loss of generality, we can set z0 = 0 and $w_0 = \infty$.

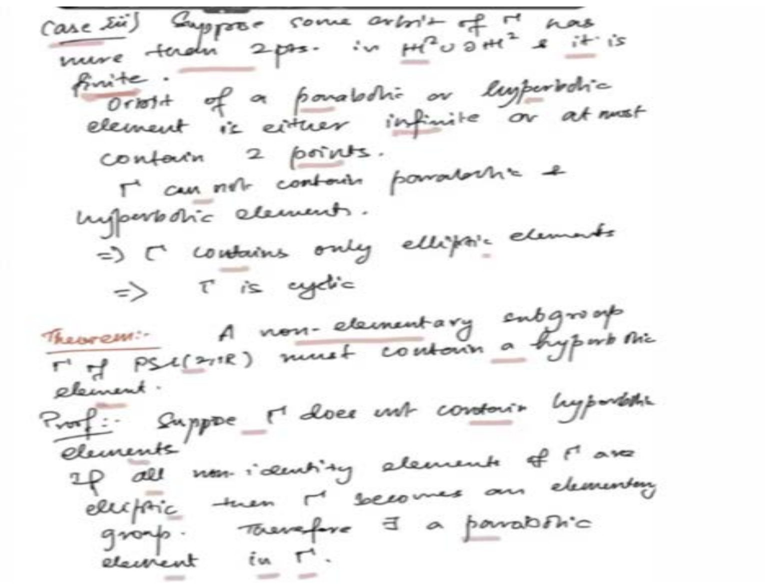
Now, let’s make an important observation: Γ cannot contain parabolic elements. A parabolic element cannot interchange the points 0 and infinity; rather, it would fix both points, which is not permissible in this context. Therefore, we conclude that Γ cannot include any parabolic elements.

This leaves us with the possibility that Γ can consist solely of elliptic or hyperbolic elements. If Γ contains only hyperbolic elements, then the fixed point set of Γ would be 0 and infinity, leading us to conclude that Γ is cyclic, this has already been established.

On the other hand, if Γ consists only of elliptic elements, it will also form a cyclic group based on our previous findings. In this scenario, we can demonstrate that Γ is conjugate to the cyclic group generated by the transformation $z \mapsto -\frac{1}{z}$.

Finally, if Γ contains both hyperbolic and elliptic elements, then it follows that Γ is conjugate to the group generated by the transformations $z \mapsto \lambda z$ and $z \mapsto -\frac{1}{z}$. Thus, for Case 2, we have reached a satisfactory conclusion.

**(Refer Slide Time: 34:42)**




Now, let's explore Case 3. Suppose we have an orbit of Γ that consists of more than two points in the upper half-plane, including the boundary, and it is, of course, finite. It's crucial to note that the orbit of a parabolic or hyperbolic element is either infinite or can contain at most two points. Given our assumption that the orbit is finite, we cannot have it being infinite. Therefore, by definition, the orbit must be finite.

This means that if the orbit of a parabolic or hyperbolic element is finite, it can only contain two points. Consequently, we can conclude that Γ cannot contain any parabolic or hyperbolic elements. Hence, it follows that Γ consists solely of elliptic elements, which implies that Γ is cyclic.

Now, let us turn our attention to the next theorem, which states the following: a non-elementary subgroup of PSL(2, R) must contain a hyperbolic element. So, how do we prove this? Suppose Γ does not contain any hyperbolic elements. If all non-identity elements of Γ are elliptic, then Γ would indeed become an elementary group, which is not permissible. Therefore, we can conclude that there must exist at least one parabolic element within Γ.

Once again, without loss of generality, we can take a conjugate of this group. Thus, we can assume that the map f(z) = z + 1 belongs to Γ. Now, let's consider any element $g(z) = \frac{az+b}{cz+d}$ that is part of Γ. If we take the isometry $f^n$ and compose it with g, we can compute the trace of this composition.

**(Refer Slide Time: 36:42)**


The corresponding matrix for $f^n \circ g$ will yield a trace that can be expressed as $\text{trace} = a + d + nc$. Since Γ does not contain any hyperbolic elements and $f^n \circ g$ belongs to Γ, it follows that the only elements present in Γ are elliptic and parabolic. Therefore, the left-hand side must be less than or equal to 2, meaning that the trace of this matrix is constrained to be less than or equal to 2, and consequently, the square of that value must be less than or equal to 4.

This leads us to the conclusion that $0 \leq a + d + nc \leq 4$ holds true for all n. The only scenario in which this condition can be satisfied is when c = 0, which implies that $g(\infty) = \infty$. Thus, infinity is fixed by all elements of Γ. This result indicates that Γ is indeed an elementary group, as the orbit of infinity reduces to a singleton set consisting solely of infinity.

This contradiction arises from our initial assumption that Γ does not contain any hyperbolic elements. Therefore, we can confidently conclude that a non-elementary subgroup of $PSL(2, R)$ must contain at least one hyperbolic element. With that, I will stop here.