An Introduction to Hyperbolic Geometry

Prof. Abhijit Pal

Department of Mathematics and Statistics

Indian Institute of Technology – Kanpur

Lecture – 17

Algebraic Structures of Fuchsian Groups: Cyclicity and Centralizers

Hello. In the previous lecture, we explored how, if two elements of PSL(2, R) or PSU(1,1) commute with each other, they must share the same fixed points. Building on that foundation, in this lecture, we will delve into some algebraic properties of Fuchsian groups. Specifically, we will demonstrate that if all the elements of a Fuchsian group share a common fixed point, then the group must be cyclic. This result will lead us to an important conclusion: a Fuchsian group cannot contain $Z \times Z$. In other words, $Z \times Z$ is not a Fuchsian group. So, let's begin!

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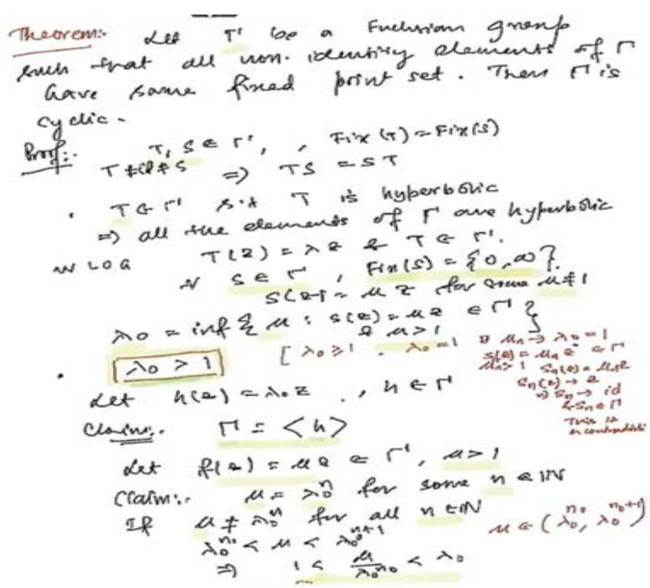

The first theorem is as follows: Let us consider a Fuchsian group, and suppose that all the non-identity elements of this Fuchsian group share the same fixed point set. We will prove that, under this condition, the Fuchsian group must be cyclic.

Let's begin by considering a Fuchsian group Γ, with the assumption that for any two elements T and S, both non-identity, their fixed point sets are the same. From a previous lemma or theorem, we know that this implies that T and S commute, i.e., $T \circ S = S \circ T$. Since T belongs to Γ, we assume that T is hyperbolic. This would imply that all elements of Γ are hyperbolic because, if we take another element S', the fixed point set of S' would also be the same as that of T, which is a two-point set, characteristic of hyperbolic transformations. Hence, every element of Γ is hyperbolic.

Now, without loss of generality, we can assume that the transformation T(z) takes the form T(z) = λ z, where T belongs to Γ. Since the fixed point set of T is {0, ∞}, it follows that for all $S \in \Gamma$, the fixed point set of S must also be {0, ∞}, and hence S(z) must take the form S(z) = μ z, for some real positive $\mu \neq 1$.

Our goal now is to prove that Γ is a cyclic group. To do this, we need to find a generator for this cyclic group. We start by considering the element T, which is hyperbolic. Let λ_0 be defined as the infimum of all such μ values, where the isometry S(z) = μ z belongs to Γ and μ > 1. Thus, λ_0 is defined as the smallest such μ.

The next claim is that $\lambda_0 > 1$. Now, let $S(z) = \lambda_0 z$. We will prove that Γ is generated by h, where $h(z) = \lambda_0 z$.

Thus, there are two key claims here: first, that $\lambda_0 > 1$, and second, that if $h(z) = \lambda_0 z$, then Γ is generated by h, making Γ a cyclic group. Now, let's address the first claim, that $\lambda_0 > 1$. First, observe that λ_0, being the infimum of all μ, where μ > 1, must satisfy $\lambda_0 \geq 1$. What we need to do next is rule out the possibility that $\lambda_0 = 1$.

Let's assume, for contradiction, that $\lambda_0 = 1$. Suppose there exists a sequence μ_n converging to λ_0, with each μ_n greater than 1, and consider the transformations $S_n(z) = \mu_n z$, where each S_n belongs to Γ. Since $\mu_n \to 1$, the transformation $S_n(z)$ converges to the identity map, i.e., $S_n(z) \to z$.

However, this creates a problem. Γ is a Fuchsian group, and by definition, it is a discrete group. But if S_n converges to the identity, we get a contradiction, as discreteness implies there cannot be a sequence of distinct elements converging to the identity. This contradiction shows that $\lambda_0 > 1$.

Now, moving to the second claim: Γ is generated by $h(z) = \lambda_0 z$. To prove this, let's consider an arbitrary element f(z) = μ z that belongs to Γ, where μ > 1. The claim is that $\mu = \lambda_0^n$ for some natural number n.

If this were not true, i.e., if μ could not be written as λ_0^n for any natural number n, then there would exist some n_0 such that μ lies strictly within the open interval $(\lambda_0^{n_0}, \lambda_0^{n_0+1})$. This implies that:

$$1 < \frac{\mu}{\lambda_0^{n_0}} < \lambda_0$$

This inequality contradicts the fact that λ_0 is the smallest such value in Γ, proving that $\mu = \lambda_0^n$ for some natural number n. Hence, Γ is generated by the transformation h(z) = λ_0 z, confirming that Γ is indeed a cyclic group.

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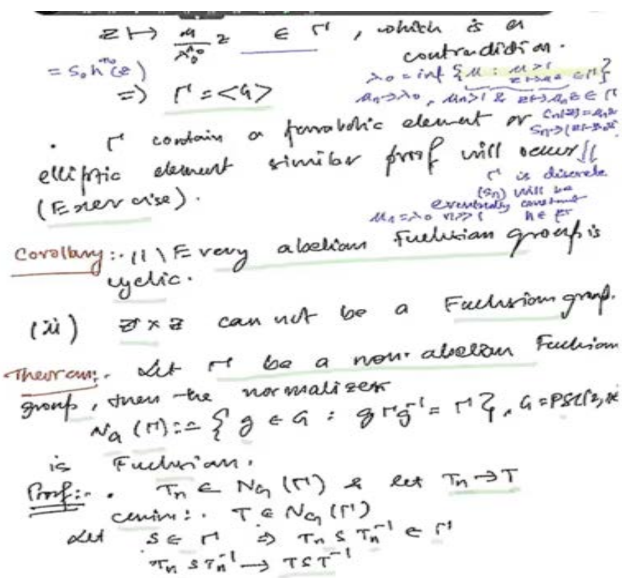

Now, let's revisit the argument and clarify it step by step. The expression $\frac{\mu}{\lambda_0^{n_0}} < \lambda_0$ essentially defines a mapping that we are working with here. So, consider the map f(z) = μ z, which we know belongs to Γ. Additionally, the map h(z) = λ_0 z also belongs to Γ, and we will explain why this is the case.

We have assumed that f(z) = μ z is in Γ, and it's important to note that h(z) = λ_0 z also belongs to Γ. But why does h belong to Γ? We need to prove this. So, let's establish this reasoning clearly.

Recall that λ_0 is the infimum of all μ, where $\mu > 1$ and the mapping $z \mapsto \mu z$ belongs to Γ. From this, we can construct a sequence μ_n that converges to λ_0, with each $\mu_n > 1$ and each mapping $S_n(z) = \mu_n z$ also belonging to Γ. Since $\mu_n \to \lambda_0$, the transformations $S_n(z)$ converge to the map $z \mapsto \lambda_0 z$.

At this point, since Γ is discrete, the sequence S_n must eventually become constant. In other words, for sufficiently large n, the sequence μ_n will stabilize, meaning that for large enough n, $\mu_n = \lambda_0$. This shows that the map $h(z) = \lambda_0 z$ indeed belongs to Γ.

Thus, we have proven that h belongs to Γ. Consequently, any power of h, whether positive or negative, also belongs to Γ. Now, consider the composition $S \circ h^{-n_0}(z)$, which is equal to $\frac{\mu}{\lambda_0^{n_0}} z$. This map belongs to Γ, as it is composed of elements from Γ.

However, here lies the contradiction: we assumed $\frac{\mu}{\lambda_0^{n_0}} < \lambda_0$, which implies that there is a value less than the infimum λ_0 such that the map $z \mapsto \frac{\mu}{\lambda_0^{n_0}} z$ belongs to Γ. But this contradicts the fact that λ_0 is the smallest such value in Γ.

Therefore, the contradiction arises due to our assumption, meaning that μ must be equal to λ_0^n for some natural number n.

Thus, we conclude that $f = h^n$ for some natural number n. Consequently, we have shown that Γ is a cyclic group generated by the element h.

Now, if Γ contains either a parabolic or elliptic element, the proof follows in a similar manner, so I will leave that as an exercise for you. As a direct corollary of this result, one can deduce that every Abelian Fuchsian group must also be cyclic. This is because, in any Abelian Fuchsian group, the elements commute with one another, which implies that they must all share the same fixed point set. By the theorem we just proved, this forces the group to be cyclic.

For example, the group $Z \times Z$ is an Abelian group but not cyclic, so it can never be a Fuchsian group. This gives us an important takeaway: $Z \times Z$ is not a Fuchsian group.

Now, let's move to the next theorem. It states that if we take a non-Abelian Fuchsian group, the normalizer of that group within $\mathrm{PSL}(2, R)$ is also Fuchsian. But first, what do we mean by the normalizer? In the group $G = \mathrm{PSL}(2, R)$, the normalizer of Γ is the set of all elements $g \in G$ such that $g\,\Gamma\,\square^{-1} = \Gamma$, meaning Γ is a normal subgroup within this normalizer.

To prove this, we begin by taking a sequence of elements T_n from the normalizer, and we assume that this sequence converges to some element T. Our first claim is that this limit T also belongs to the normalizer.

Let's take any element $S \in \Gamma$, and consider that each T_n belongs to the normalizer of Γ. This means that $T_n S T_n^{-1} \in \Gamma$, for all T_n. Now, since $T_n \to T$, we can conclude that $T_n S T_n^{-1} \to T S T^{-1}$, and it's straightforward to verify that this implies $T S T^{-1} \in \Gamma$, proving that T is indeed in the normalizer.

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So, we have $T_n S T_n^{-1} \in \Gamma$, and since we started with Γ as a Fuchsian group, this implies that the sequence $T_n S T_n^{-1}$ will eventually become constant. Consequently, we have the following: $T_n S T_n^{-1} = T S T^{-1}$ for sufficiently large n. Given that $T_n S T_n^{-1} \in \Gamma$, it follows that $T S T^{-1}$ must also belong to Γ.

Now, since $S \in \Gamma$, was arbitrary, we have shown that $T S T^{-1} \in \Gamma$, which implies that T is in the normalizer of Γ. Therefore, we have successfully proved that if a sequence T_n from the normalizer converges to T, then T also belongs to the normalizer.

To establish that the normalizer of Γ is a Fuchsian group, we must further demonstrate that the sequence T_n is eventually constant, i.e., $T_n = T$ for all sufficiently large n. Suppose, for contradiction, that this is not the case, and that T_n is a distinct sequence. While T_n converges to T, the difference $P_n = T^{-1}T_n$ must converge to the identity. Since both T and T_n belong to the normalizer, it follows that $P_n = T^{-1}T_n$ ∈ the normalizer as well, and $P_n \rightarrow$ Identity by our assumption.

However, we have also assumed that T_n is not eventually constant, meaning $P_n \neq Identity$ for all n. Now, take any element $S \in \Gamma$. Since $P_n \in$ the normalizer, it follows that $P_n S P_n^{-1} \in \Gamma$. Moreover, since $P_n \rightarrow Identity$, we have $P_n S P_n^{-1} \rightarrow S$. But because Γ is a Fuchsian group, which is discrete, this convergence implies that $P_n S P_n^{-1} = S$ for large n, resulting in a contradiction.

Therefore, our assumption that $P_n \neq Identity$ was incorrect, and we conclude that the sequence T_n must be eventually constant, i.e., $T_n = T$ for all large n. This completes the proof that the normalizer of a non-Abelian Fuchsian group is itself a Fuchsian group.

Therefore, this sequence P_n will eventually become constant. Consequently, for all sufficiently large n, we have $P_n S P_n^{-1} = S$, which demonstrates that the fixed points of S are the same as those of P_n for all $n \geq n_S$. Now, let us consider another element $S' \in \Gamma$. Similarly, for all $n \geq n_{S'}$, the fixed points of S' will also coincide with those of P_n.

Now, let us take $n \geq \max(n_S, n_{S'})$. In this case, the fixed points of S are equal to the fixed points of P_n, and the fixed points of P_n are equal to the fixed points of S'. Thus, the fixed points of S and S' are the same. By a theorem we established in the previous class, this implies that S and S' commute with each other.

Since S and S' were arbitrary elements of Γ, we have proven that all elements of Γ commute with one another. This proves that Γ is abelian. However, we began by assuming that Γ is nonabelian, leading to a contradiction. The contradiction arises from the assumption that T_n is not an eventually constant sequence.

Therefore, the sequence T_n must be eventually constant, meaning $T_n = T$ for all sufficiently large n. This ultimately implies that the normalizer of Γ is indeed a Fuchsian group.